Identifying the Most Connected Vertices in Hidden Bipartite Graphs Using Group Testing

Jianguo Wang, Eric Lo, and Man Lung Yiu

Abstract—A graph is called hidden if the edges are not explicitly given and edge probe tests are required to detect the presence of edges. This paper studies the $k$ most connected vertices ($k$MCV) problem on hidden bipartite graphs, which has applications in spatial databases, graph databases, and bioinformatics. It presents an algorithm, namely, GMCV, that adaptively leverages group testing to solve the $k$MCV problem.

1 INTRODUCTION

A graph is called hidden if the edges are not explicitly given and edge probe tests are required to detect the presence of edges [18]. Recently, Tao et al. [29], [28] studied the $k$ most connected vertices ($k$MCV) problem on hidden bipartite graphs. Specifically, given a hidden bipartite graph $G$ with two independent vertex sets $B$ (black vertex set) and $W$ (white vertex set), the $k$MCV problem is to find the top $k$ vertices in $B$ that have the maximum degree. Fig. 1 shows a hidden bipartite graph $G$, where $B = \{b_1, b_2\}$ and $W = \{w_1, w_2, \ldots, w_8\}$. The $k$MCV aims to identify the vertex $b_1$ because it has the largest degree. The problem is trivial on conventional bipartite graphs but not in the case of hidden graphs because edge probe tests are usually expensive operations (e.g., biological experiments, graph operations). The applications of finding the $k$MCV on a hidden bipartite graph include distance join on road networks, bioinformatics, and graph pattern matching [29], [28].

Example 1. Distance join on road networks. Let $B$ and $W$ be the hotel set and scenic spot set, which constitute a bipartite graph $G(B,W)$. A hotel $b \in B$ and a scenic spot $w \in W$ has an edge if their distance is less than a threshold $\theta_{\text{dist}}$, for example, 5 km, where the distances are shortest path distances. Therefore, the $k$MCV problem could help discover the most convenient hotels. While the edges on $G$ are not given initially, a shortest path algorithm could be executed to detect their presence. Fig. 2 shows a road network. Fig. 1 shows the hidden graph representation of distance join in Fig. 2, using $\theta_{\text{dist}} = 5$. To detect whether hotel $b_1$ and scenic spot $w_2$ have an edge connecting in Fig. 1, we can run a shortest path algorithm as the edge probe test to find the shortest path between $b_1$ and $w_2$ in Fig. 2. In this example, the shortest path distance between $b_1$ and $w_2$ is 2; thus, after the execution of the shortest path algorithm, the edge that connects $b_1$ and $w_2$ in Fig. 1 becomes explicit. Shortest path queries on large graphs are usually computationally expensive [30]. Therefore, the goal of $k$MCV is to find the answer using an efficient strategy.

Example 2. Bioinformatics. In bioinformatics, interactions between proteins are often represented as graphs. Specifically, the interactions between bait proteins ($B$) and prey proteins ($W$) could form a hidden bipartite graph $G(B,W)$ [21], [22]. An edge $(b,w)$ represents a bait protein $b$ interacts with a prey protein $w$, and this interaction could be discovered by carrying out an edge probe test in the form of a biological experiment, which may take hours or days [17]. The $k$MCV problem is to find the most active proteins. And it would be beneficial if there is a way to get the answer efficiently.

Example 3. Graph pattern matching. Applications like drug discovery often need to identify the graph patterns that match the most number of data graphs [29], [28]. The discovery process usually involves testing whether a graph pattern $b$ is a sub/supergraph of a data graph $w$. An edge is present if such a containment relationship exists between $b$ and $w$. Such information, however, remains hidden unless an explicit sub/supergraph containment test is carried out. Unfortunately, such testing is known to be expensive, for example, a subgraph isomorphism test is NP-complete [9], [27]. Therefore, it is necessary to devise an efficient algorithm for the $k$MCV problem to speed up the drug discovery process.

As the pioneering work, Tao et al. [29], [28] developed an algorithm, SOE, to solve the $k$MCV problem. SOE is based

1. Actually, Tao et al. [29], [28] proposed two algorithms: Sample-and-Set (SS) and Switch-on-Empty. Since SOE outperforms SS in both theory and in practice, we, therefore, focus on SOE only.
on 2-vertex edge probe testing, or simply 2-vertex testing [7], i.e., each edge probe test \(Q(b, w)\) takes as inputs one black vertex \(b \in B\) and one white vertex \(w \in W\), and returns 1 if \(b\) and \(w\) possess an edge in the hidden bipartite graph \(G\) and 0 otherwise. In many applications [11], [13], [32], [7], the more general vertex-group edge probe testing is used as a replacement of the 2-vertex model. Specifically, a vertex-group edge probe test, or simply, a group test, takes as inputs one black vertex \(b \in B\) and a group of white vertices \(W \subseteq W\), denoted as \(Q(b, W)\), and returns 1 if there exists at least one white vertex \(w \in W\) possessing an edge with \(b\) in the hidden graph \(G\) and 0 otherwise. We observe that such a test model is also applicable to the \(k\)MVC problem (in above applications):

- In the distance join application, if a road network index [19], [25], [31] is available, a group test \(Q(b, W)\) can be implemented by asking the road network index the nearest neighbor of a vertex \(b\) (denoted as \(w_{nn}\)) in a given group of vertices \(W\). If \(\text{dist}(b, w_{nn}) > \theta_{\text{dist}}\), we learn that all vertices in \(W\) are beyond \(\theta_{\text{dist}}\) of \(b\); therefore, none of the vertices in the group \(W\) connects with \(b\) in the hidden graph, i.e., \(Q(b, W) = 0\). Otherwise, we get \(Q(b, W) = 1\).

- In bioinformatics, the literature does show that many biological experiments can be set up to tell whether there are reactions between a protein \(b\) and a set of proteins \(W\) [22], [7].

- In the graph matching application, a graph index \(I_W\) (e.g., FG-index [9], cIndex [8], GPTree [33]) can be built on a set of data graphs \(W\). A group test \(Q(b, W)\) can be regarded as a pattern query \(b\) on the set \(W \subseteq W\) to check whether there exists a data graph \(w \in W\) such that \(b\) and \(w\) satisfy the containment relationship. If yes, then \(Q(b, W) = 1\), and \(Q(b, W) = 0\) otherwise. Notice that \(W\) corresponds to a particular subtree of the index \(I_W\). Thus, the group test can be implemented by issuing \(b\) as a graph query to the corresponding subtree of \(I_W\).

Table 1 gives a summary of how the above applications associated with the \(k\)MVC problem in the context of group testing.

The applicability of group testing on the \(k\)MVC problem raises a very interesting research question: Can we leverage group testing to solve the \(k\)MVC problem more efficiently? Specifically, a group test \(Q(b, W)\) returning 0 is equivalent to revealing many hidden edges in a row: \(Q(b, w_1) = 0, Q(b, w_2) = 0, \ldots, Q(b, w_k) = 0\), for all \(w_i \in W\). If an algorithm can leverage it smartly and correctly, the number of tests can be significantly reduced. However, although the use of group test may reduce the number of tests in solving the \(k\)MVC problem, we have to ensure that the actual cost of solving the \(k\)MVC problem can essentially be reduced. That is because the cost (e.g., monetary cost, running time) of a group test execution, in which we call that as external cost, may be more than the external cost of a 2-vertex edge probe test execution, because the former may take more than two white vertices as input. Fortunately, in all of the applications that we concern, the external cost of a group test is indeed sublinear to or even independent of the input size. For example, in the distance join application and the graph pattern matching application, it has been shown that the external cost (running time) of checking the nearest neighbor between a vertex \(b\) and a set of vertices \(W\) using a road network index and the external cost (running time) of checking the containment relationship between a pattern \(b\) and a set of data graphs \(W\) using a graph index are sublinear to the size of \(W\) [19], [25], [31], [9], [8], [33], because of the indices’ high pruning effectiveness. In bioinformatics, it is a well-known fact that the external cost of a group test, no matter in terms of the monetary cost (e.g., the cost of the chemical used) or the time to finish an experiment, is independent of the number of input chemicals involved in the experiment [4], [5], [3], [15].

To leverage group testing, we have to design the algorithm carefully because it is tricky to determine the

![Fig. 1. A (hidden) bipartite graph \(G(B, W)\); edges are not explicitly given.](image1)

![Fig. 2. Example of a road network.](image2)

**Table 1** Applications That Can Apply Group Testing

<table>
<thead>
<tr>
<th>Application</th>
<th>Meaning of the Black Vertex Set (B)</th>
<th>Meaning of the White Vertex Set (W)</th>
<th>Meaning of a Hidden Edge ((b, w))</th>
<th>Meaning of a Group Test (Q(b, W))</th>
<th>External Cost of a Group Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance join</td>
<td>locations (hotel sets)</td>
<td>locations (spot sets)</td>
<td>the distance of (b) and (w) is less than a threshold (\theta_{\text{dist}})</td>
<td>Run shortest path algorithm: the distance of (b) and at least one vertex in (W) is less than (\theta_{\text{dist}})</td>
<td>sub-linear to group size</td>
</tr>
<tr>
<td>Bioinformatics</td>
<td>bait proteins</td>
<td>prey proteins</td>
<td>(b) interacts with (w)</td>
<td>Conduct biological experiment: (b) interacts with at least one vertex in (W)</td>
<td>constant</td>
</tr>
<tr>
<td>Graph pattern matching</td>
<td>data graphs</td>
<td>data graphs</td>
<td>(b) is a sub/super-graph of (w)</td>
<td>Query on graph index: (b) is a sub/super-graph of at least one data graph in (W)</td>
<td>sub-linear to group size</td>
</tr>
</tbody>
</table>
input size of the white vertex set, i.e., |W|, for each group testing. Even though the external cost of a group test is usually sublinear to or independent of the group size, we still should not deliberately include a lot of vertices in each group test because that would increase the chance of the testing result being 1. Such a result is actually not informative because it does not reveal any hidden edge between any pair of black vertex and white vertex. However, if a very small group size is used, the power of group testing may not be well exploited. Therefore, it is challenging to leverage the group test model in a productive manner.

Based on the discussions above, we propose an algorithm, GMCV, that leverages group testing to solve the \( k \)-MCV problem. Note that if the group size \( |W| \) is always set to 1, a group test is the same as 2-vertex testing. Therefore, GMCV is more general than SOE. GMCV adaptively controls the group sizes based on the data characteristics during execution. For applications like distance join and graph pattern matching, GMCV can be regarded as a usual computer algorithm that aims to solve the \( k \)-MCV problem efficiently. For applications like bioinformatics, GMCV can serve as an offline human-involving tool like [23] that assists human (scientists) in scheduling their actions (experiments) using the least amount of external resources. Specifically, GMCV can suggest to a scientist what experiment should be done next after finishing the current experiment (which may take days).

The rest of the paper is organized as follows: We review the related work in Section 2. We formally define the problem in Section 3. Then, we present the technical contributions in the following order:

- First, we present the details of GMCV, a more general algorithm for solving the \( k \)-MCV problem, in Section 4.
- Then, we present cost models of GMCV and SOE in Section 5. Notice that the total external testing cost of an execution of GMCV not only depends on
  - the number of group tests executed, but also
  - the input size to each group test and
  - the implementation of the group test.
- For example, the time complexity of a group test in the distance join application is sublinear to the input group size. However, in bioinformatics, a group testing is an actual (chemical/biological) experiment, in which its cost (running time/monetary cost) is independent of the group size.
- Finally, we experimentally evaluate GMCV in Section 6. The evaluation is done on both real-life data sets and synthetic data sets. The experimental results show that GMCV is a good general alternative to SOE.

After presenting the above contributions, we conclude the paper in Section 7. Table 2 summarizes the symbols used in the subsequent sections.

### 2 Related Work

Hidden graph has been an active research topic in the computing theory community [18], [4], [3]. Applications of hidden graph are mostly bioinformatics related. One branch of hidden graph research is graph testing: Given a hidden graph \( \mathcal{G} \), the objective is to test whether \( \mathcal{G} \) possesses a certain property (e.g., \( k \)-colorable [16]) using a minimal number of edge probe tests (e.g., biological experiments). Another branch of hidden graph research is graph learning: Given a hidden graph \( \mathcal{G} \), the objective is to reconstruct the whole graph using a minimal number of edge probe tests [18], [4], [3], [7], [15]. As argued by Tao et al. [29], [28], the \( k \)-MCV problem is different from those work because it neither tests the possession of any property of the hidden graph, nor reconstructs the whole graph.

The works above are all based on the 2-vertex edge probe test model [7]. They assume that the cost of a 2-vertex test is a constant. So, the costs of those algorithms are analyzed based on the number of tests they invoked. Thus, it is natural that those works focus on reducing the number of tests. Recently, the more general vertex-group edge probe test model is used in both graph testing and graph learning [13], [32], [7], because in those applications the cost of a group test is independent of the group size. This paper aims to investigate the use of group testing in solving the \( k \)-MCV problem.

Comparing with SOE [29], [28], the use of group testing raises at least two new technical aspects: 1) In terms of algorithm design, a \( k \)-MCV algorithm that exploits the group test model has to determine the group size carefully, in which algorithms that based on the 2-vertex model do not. 2) In terms of solution analysis, the analysis has to base on the external testing cost, which depends on a) the number of executed group tests, b) the group size, and c) the cost function of various group testing implementations.

### 3 Problem Definition

We formally define the \( k \)-MCV problem under the group testing model.

Let \( \mathcal{G} = (B, W, E) \) be a bipartite graph, where \( B \) is a set of black vertices, \( W \) is a set of white vertices, and \( E \) is a set of edges connecting vertices in \( B \) and \( W \). \( \mathcal{G} \) is hidden if \( E \) is not explicitly given. An edge probe test, or simply a test, can be carried out to detect the presence of edges.

**Definition 1 (2-vertex testing).** An edge probe test \( Q(b, w) \) is called 2-vertex testing if it asks whether a black vertex \( b \in B \) connects with a white vertex \( w \in W \):
Let \( Q(b, w) = \begin{cases} 1 & \text{if } (b, w) \in \mathcal{E} \\ 0 & \text{if } (b, w) \notin \mathcal{E}. \end{cases} \)

The 2-vertex testing method is used by SOE [29], [28]. As mentioned earlier, in many applications, for example, distance join, protein-protein interaction, we can test a group of vertices together.

**Definition 2 (Group testing).** Let \( W \) be a group of white vertices; an edge probe test \( Q(b, W) \) is called group testing if it asks whether a black vertex \( b \in B \) connects with at least one white vertex \( w \in W \):

\[
Q(b, W) = \begin{cases} 1 & \text{if } \exists w \in W, (b, w) \in \mathcal{E} \\ 0 & \text{if } \forall w \in W, (b, w) \notin \mathcal{E}. \end{cases}
\]

When \(|W| = 1\), group testing is the same as 2-vertex testing. Hence, 2-vertex testing is a special case of group testing. Depending on the actual applications, the cost of group testing may or may not depend on the input sizes.

**Definition 3 (External testing cost \( \beta \)).** Let \( Q(b, W) \) be a group test, the external cost (e.g., monetary cost, running time) of carrying out such a test is denoted as \( \beta(b, W) \). For simplicity, we represent \( \beta(b, W) \) using the input size, i.e., \( \beta(|W|) \).

**Definition 4 (kMVC).** Given a hidden graph \( G = (B, W, \mathcal{E}) \), a user-threshold \( k \) identify a minimal result set \( R \subseteq B \) such that

1. \(|R| \geq k\); and
2. \( d_i > d_j \) for any \( b_i \in R \) and \( b_j \in B \setminus R \), where \( d_i \) is the degree of \( b_i \).

The goal of this paper is to minimize the total external testing cost of solving the kMVC problem using group testing. For ease of presentation, we assume that there is no tie on the vertex’s degree such that there is exactly \( k \) vertices in the result set \( R \). Our techniques can be easily extended to handle the tie case.

## 4 Algorithm GMCV

In this section, we present our GMCV algorithm that solves the kMVC problem by the use of group testing, which aims to reduce the external testing cost. We first put down the relevant definitions.

**Definition 5 (Hidden vertex and hidden edge).** For a vertex pair \((b, w)\) where \( b \in B \) and \( w \in W \), \( w \) is a hidden vertex of \( b \) if the connection between \( b \) and \( w \) in the hidden graph \( G \) is unknown. If \( w \) is a hidden vertex of \( b \), then \((b, w)\) is a hidden edge.

**Definition 6 (Solid and empty vertex).** For a vertex pair \((b, w)\) where \( b \in B \) and \( w \in W \), if \((b, w)\) \( \in \mathcal{E} \), then \( w \) is a solid vertex of \( b \); otherwise, \( w \) is an empty vertex of \( b \).

**Definition 7 (Completed).** A black vertex \( b \) is completed if it has no hidden edges.

GMCV finds the top \( k \) black vertices with the highest degree in iterations. In each iteration, it examines the black vertices \( b_1, b_2, \ldots, b_{|B|} \) in \( B \) one by one. For a black vertex \( b_i \), some group tests are carried out between it and some white vertices \( W \subseteq W \) to tighten the degree bounds of \( b_i \), except when \( b_i \) is completed, or when \( b_i \) is deliberately skipped in that iteration because of the poor chance for \( b_i \) being in the final result (more on this later). After one iteration, another iteration starts and the black vertices \( b_1, b_2, \ldots, b_{|B|} \) in \( B \) are examined once again. Similar to most top \( k \) processing algorithms (e.g., [14], [20]), GMCV maintains the degree upper bound (denoted as \( b_{\text{maxDeg}} \)) and lower bound (denoted as \( b_{\text{minDeg}} \)) of each black vertex \( b_i \in B \) throughout the execution and stops when the following condition holds:

**Property 1 (Stop condition).** Let \( \tau \) be the \( k \)th largest degree in the result set \( R \), and \( \mu \) be the maximum degree upper bound of vertices not in \( R \), GMCV can stop and return \( R \) when \( \tau > \mu \).

With the skeleton of GMCV in place, we study the following research issues:

**R1.** In an iteration, when a black vertex \( b_i \) is being examined by GMCV, how to leverage group testing to refine \( b_i \)'s degree bounds? Specific issues include:

a. How to determine the group of white vertices that should be tested with \( b_i \) ?

b. When shall GMCV stop examining \( b_i \) in this iteration and switch to another black vertex?

**R2.** Black vertices with low degrees are unlikely to be in the top \( k \) result set \( R \), thus, the question is: How to avoid unnecessary testing for low-degree vertices?

### 4.1 Dealing with Research Issue R1

GMCV follows the “switch-on-empty” (SOE) principle [29], [28] to deal with research issue R1b. Within an iteration, it continues to work on \( b_i \) until a test returns “empty,” i.e., \( Q(b_i, W) = 0 \), or \( b_i \) becomes completed. For a black vertex \( b_i \), let \( W_{\text{CUR}} \) be the set of white vertices that \( b_i \) is going to carry out group testing with, and \( W_{\text{PRE}} \) be the previous set of white vertices that \( b_i \) carried out group testing with.

To deal with research issue R1a, GMCV adaptively identifies \( W_{\text{CUR}} \) based on \( W_{\text{PRE}} \) and the two possible “states” associated with \( b_i \): expanding, and identifying. Initially, the state of every \( b_i \in B \) is expanding, \( W_{\text{PRE}} \) is set to empty, and \( W_{\text{CUR}} \) is set to one random white vertex. For other cases (except initialization), \( W_{\text{CUR}} \) is determined as follows:

When \( b_i \) is in the expanding state, the objective of group testing between \( b_i \) and a set of white vertices is to reveal as many hidden vertices of \( b_i \) as possible:

- **Case EXP-(a):** If \( Q(b_i, W_{\text{PRE}}) = 0 \), the number of white vertices that should be involved in the upcoming group test, denoted as \( |W_{\text{CUR}}| \), is set as twice the size of \( |W_{\text{PRE}}| \), i.e., \( |W_{\text{CUR}}| = 2 \cdot |W_{\text{PRE}}| \).

This is called the doubling strategy, which is commonly used in problems to dynamically adjust the value of some unknown parameters [6], [10]. The rationale is that if \( Q(b_i, W_{\text{PRE}}) = 0 \), it implies \( b_i \) might have a low degree. Thus, GMCV can aim higher in this test—set \( b_i \) to test with a larger group of white vertices and hope that can reveal even more hidden

2. In fact, other strategies such as multiplying the group size by 3 [12] or 4 [26] do exist. However, the literature does emphasis on the doubling strategy because of its stableness.
vertices of $b_i$. The set $W_{CUR}$ is then randomly chosen from $b_i$’s hidden vertices.

- **Case EXP-(b):** If $Q(b_i, W_{PRE}) = 1$ and $|W_{PRE}| = 1$, it means $b_i$ is a potentially high-degree vertex, so GMCV keeps $|W_{CUR}| = 1$.

- **Case EXP-(c):** If $Q(b_i, W_{PRE}) = 1$ and $|W_{PRE}| > 1$, it implies that GMCV were too aggressive in the previous group test. In this case, $b_i$ enters the identifying state. When $b_i$ is in the identifying state, the objective of group testing becomes to identify at least one of the solid vertices in $W_{PRE}$ of $b_i$. Therefore,

  - **Case IDF-(a):** If $|W_{PRE}| > 1$ and $Q(b_i, W_{PRE}) = 1$, GMCV will devote some more tests to locate the white solid vertex in $W_{PRE}$. To do so, GMCV splits $W_{PRE}$ into two halves: $W_{L}^{PRE}$ and $W_{R}^{PRE}$, and sets $W_{CUR}$ to be $W_{L}^{PRE}$ and saves $W_{R}^{PRE}$ as an unexplored set $W_{U}$.

  - **Case IDF-(b):** If $|W_{PRE}| = 1$ and $Q(b_i, W_{PRE}) = 1$, that means a white solid vertex of $b_i$ in $W_{PRE}$ has been identified; in this case, GMCV resets $b_i$’s state back to the expanding state.

  - **Case IDF-(c):** If $Q(b_i, W_{PRE}) = 0$, GMCV explores the unexplored set by setting $W_{CUR}$ to be $W_{U}$, but the test result of $Q(b_i, W_{CUR})$ is explicitly encoded as 1.

After identifying $W_{CUR}$, GMCV then executes such a group testing $Q(b_i, W_{CUR})$. As mentioned, GMCV follows the switch-on-empty principle, so it may carry out a number of group tests, between $b_i$ and a number of groups of white vertices, before it switches to another black vertex in the same iteration.

Fig. 4 shows an example that illustrates some of the cases above. The corresponding input hidden graph is shown in Fig. 3. In the first iteration, $b_1$ is first considered and $W_{CUR} = \{w_1\}$ (a random white vertex) (Iteration 1a). After the first group test $Q(b_1, W_{CUR})$, it is found that $w_1$ is a solid vertex of $b_1$. This falls into [Case EXP-(b)] described above, resulting $W_{CUR}$ is set to another random vertex $w_2$ (Iteration 1b). After the next group test $Q(b_1, W_{CUR})$, it is found that $w_2$ is an empty vertex of $b_1$. So, GMCV follows the switch-on-empty principle and considers $b_2$ (Iteration 1c). Since $b_2$ is first visited by GMCV, its $W_{CUR}$ is set as $\{w_1\}$, like what happened to $b_1$.

---

**Fig. 4.** Running example.
After the group test $Q(b_j, W^{\text{CUR}})$, it is found that $w_1$ is an empty vertex of $b_j$. Therefore, GMCV has to switch to another vertex, leading to Iteration 2, which considers $b_j$ again (Iteration 2a). At that point, for $b_j$, $W^{\text{PRE}} = \{w_2\}$ (refer to Iteration 1b), so it falls into [Case EXP-(a)] described above, causing the size of $W^{\text{CUR}}$ to be doubled (Iteration 2a). After the group test $Q(b_j, W^{\text{CUR}})$, it is found that $w_3$, or $w_4$, or both are solid vertices of $b_j$, so it falls into [Case EXP-(c)] described above, $b_j$’s state is thereby switched to identifying (Iteration 2b). At that point, for $b_j$, $W^{\text{PRE}} = \{w_3, w_4\}$, so it falls into [Case IDF-(a)] described above, resulting $W^{\text{CUR}}$ is set as $\{w_3\}$. After the group test $Q(b_j, W^{\text{CUR}})$, it is found that $w_3$ is an empty vertex of $b_j$ (which then also implies $w_4$ is a solid vertex of $b_j$), which triggers GMCV to switch to $b_j$ (Iteration 2c). After the group test $Q(b_j, W^{\text{CUR}})$, it is found that both $w_2$ and $w_3$ are empty vertices of $b_j$, making GMCV switches to $b_1$ again (Iteration 3a). By that time, although $Q(b_1, W^{\text{CUR}})$ supposes to test with $W^{\text{CUR}}$, it falls into the case of [Case IDF-(c)], in which the test result is already encoded as 1 without even testing. So, after that, GMCV continues testing between $b_1$ and another white vertex $w_5$ (Iteration 3b), and the process goes on until the stopping condition (Property 1) holds.

4.2 Dealing with Research Issue R2

For each black vertex $b_j \notin R$, the “necessary” tests are to reduce its degree upper bound, until below $\tau$. In other words, it should not have any further testing once its degree upper bound below $\tau$, as it is not part of the result set. However, the value of $\tau$ is unknown in advance; therefore, $b_j$ may get redundant tests even if $b_j.\maxDeg$ is really less than $\tau$ during the execution.

Thus, the question is, for any $b_j \notin R$ (i.e., low-degree vertex), how to prevent it from any further unnecessary testing even though $\tau$ is unknown beforehand? In other words, how to guarantee for any $b_j \notin R$, it does not have any unnecessary testing once $b_j.\maxDeg < \tau$?

GMCV employs a skipping policy to achieve the goal. If $Q(b_j, W^{\text{CUR}}) = 0$, then $b_j$ is skipped for a skip factor of $|W^{\text{CUR}}| - 1$ iterations. For example, if at iteration $i$, $Q(b_j, \{w_1, w_2, w_3\}) = 0$, then GMCV skips $b_j$ in the iterations $i + 1$ and $i + 2$. In Theorem 1 (Section 4.3), we will show that with our skipping policy, vertices not in the result set do not have unnecessary testing. Then, we will show in Lemma 4 (Section 4.3) that the skip factor $|W^{\text{CUR}}| - 1$ is the optimal one among all the possible choices, so GMCV will use that as the skip factor. In the following, we first present the algorithm GMCV.

4.3 Algorithm: GMCV

The pseudocode of GMCV is listed below. It is self-explanatory. It employs a skip factor of $|W^{\text{CUR}}| - 1$. Each black vertex $b_j$ is associated with a field skip, which gets incremented whenever a group test has identified a group of $b_j$’s empty vertices in a single group test, resulting in the skipping of processing $b_j$ in a number of subsequent iterations.

**Algorithm GMCV**

**Input**

$G(B, W)$: Hidden bipartite graph; $k$: User-threshold

**Output**

$R$: $k$ black vertices that have the maximum degree

1. $\tau$: the degree of the $k$-th ranked vertex in $R$
2. $\mu$: the maximum degree upper bound for those vertices not in $R$, i.e., $\max_{b \in B}.\maxDeg$
3. $R$ is initialized to $k$ dummy vertices with degree $−1$
4. for each $b \in B$ do
5.   $b.\minDeg \leftarrow 0$ /*degree lower bound*/
6.   $b.\maxDeg \leftarrow |W|$ /*degree upper bound*/
7.   $b.\skip \leftarrow 0$ /*implement the skip policy*/
8. repeat
9.   /*start an iteration*/
10.   for each $b \in B$ do
11.     if $b$ is completed then continue
12.     if $b.\skip > 0$ then /*skip policy*/
13.        $b.\skip \leftarrow b.\skip - 1$
14.     continue
15.     find a group of white vertices $W^{\text{CUR}}$ to test /*Section 4.1*/
16.     if $Q(b, W^{\text{CUR}}) = 0$ then /*external testing*/
17.        $b.\maxDeg \leftarrow b.\maxDeg - |W^{\text{CUR}}|
18.        $b.\skip \leftarrow b.\skip + (|W^{\text{CUR}}| - 1)$
19.     else
20.        if $|W^{\text{CUR}}| = 1$ then
21.            $b.\minDeg \leftarrow b.\minDeg + 1$
22.     goto line 10
23.   let $C$ be the completed vertices in this iteration
24.   $R \leftarrow R \cup C$
25.   update $\tau$ /*$k$-th largest degree in $R$*/
26.   $R \leftarrow \{b \in R : d_i \geq \tau\}$ /*update the result set $R$*/
27.   update $\mu$ /*upper-bound score of vertices not in $R$*/
28. until $\mu < \tau$

Table 3 shows the detailed execution steps of GMCV in finding the 1MCV of the hidden graph presented in Fig. 3. The final $\tau$ value is 10, which is the degree of $b_1$ but is unknown till the end of GMCV. After the fourth iteration, $b_2.\maxDeg = 9$, which is below $\tau$. Since then, $b_2$ is skipped for any further tests, until the end of GMCV.

**Lemma 1.** GMCV correctly reports the results, i.e., black vertices with top $k$ maximum degrees.

**Proof.** The stopping condition $\mu < \tau$ (Property 1) guarantees that for any vertices not in $R$ will not have a higher degree than those in $R$.

**Theorem 1.** In GMCV, a black vertex $b_j \notin R$ stops any further testing, once its degree upper bound is just smaller than the final $\tau$.

**Proof.** The statement is equivalent to, any black vertex $b_j \notin R$ stops for any further testing once the number of empty vertices it has detected is greater than or equal to $|W| - (\tau - 1)$. Let $\theta = |W| - (\tau - 1)$.

Formally, let $E_{b_j}$ be the number of empty vertices detected with $b_j$ during GMCV, then $E_{b_j}$ is increasing during the execution of the algorithm. Let $E^n_{b_j}$ be the value of $E_{b_j}$ after the $n$th change of $E_{b_j}$. (Thus, $E^n_{b_j} \leq E^{n+1}_{b_j}$).
Let $\mathcal{E}_{bj}$ be the value of $\mathcal{E}_b$ of the last change of $\mathcal{E}_b$ before GMCV terminates, we have 1) $\mathcal{E}_{bj} \geq \theta$ and 2) $\mathcal{E}_{bj}^{-1} < \theta$.

We prove 1 by contradiction. At the end of GMCV, if $\mathcal{E}_b < \theta$ (i.e., $\mathcal{E}_{bj}^{-1} < \theta$), we have $b_{j,\text{maxDeg}} = |W| - \mathcal{E}_{bj} > |W| - \theta = \tau - 1$. In other words, $b_{j,\text{maxDeg}} \geq \tau$. According to the stop condition of GMCV (Property 1), $\mu < \tau$, where $\mu$ is the maximum degree upper bound of vertices not in $R$, meaning that $b_{j,\text{maxDeg}} < \tau$, which is a contradiction.

Next, we will prove 2 $\mathcal{E}_{bj}^{-1} < \theta$ by contradiction. Let us assume

$$\mathcal{E}_{bj}^{-1} \geq \theta. \tag{1}$$

We state the supplementary Lemmas 2 and 3, which are proved in the appendix, which can be found on the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TKDE.2012.178.

**Lemma 2.** Let the $(z-1)$th change of $\mathcal{E}_b$ value occurs at the end of iteration-$I$ of GMCV, if $b_j,\text{skip} = 0$, then iteration-$I$ is the last iteration of GMCV.

**Lemma 3.** Let the $(z-1)$th change of $\mathcal{E}_b$ value occurs at the end of iteration-$I$ of GMCV, if $b_j,\text{skip} > 0$, then at the end of the iteration-$(I + b_j,\text{skip})$, GMCV must have terminated.

With Lemma 2 proven, it implies that the $(z-1)$th change of $\mathcal{E}_b$: is the last change of $\mathcal{E}_b$, which contradicts the fact that $\mathcal{E}_{bj}^{-1}$ is the last change of $\mathcal{E}_b$.

With Lemma 3 proven, and together with the fact that the value of $\mathcal{E}_b$ does not change between iteration-$I$ and iteration-$(I + b_j,\text{skip})$ (because by that time $b_j,\text{skip} > 0$ and thus $b_j$ is skipped), so the value of $\mathcal{E}_b$ at iteration-$(I + b_j,\text{skip})$ is equal to the value of $\mathcal{E}_b$ at the end of the iteration-$I$, which is equal to $\mathcal{E}_{bj}^{-1}$. So, if we can prove that GMCV has terminated by that time, it implies that $\mathcal{E}_{bj}^{-1}$ is the value of $\mathcal{E}_b$ before GMCV terminates, which contradicts the fact that $\mathcal{E}_{bj}^{-1}$ is the last change of $\mathcal{E}_b$.

With Lemmas 2 and 3 proven, we can conclude that the assumption $\mathcal{E}_{bj}^{-1} \geq \theta$ is false and the proof is completed.

**Lemma 4.** Let $C_s$ be the external testing cost of our solution with skip factor of $s$, then for any $s$, $C_{\text{OPT}} \leq C_s$.

**Proof.** First, each $b \in B$ has a sequence of group tests and stops when the stopping condition (Property 1) is met. Obviously, for a $b_j$ in the final result set $R$, its whole sequence of group tests must be carried out. So, we care about only those $b_j$ not in the final result set $R$. For $b_j \notin R$, its degree upper bound, denoted as $b_{j,\text{maxDeg}}$, gets reduced along the iterations when more tests are done. Its corresponding aggregated external testing cost is the minimum if its test sequence stops once $b_{j,\text{maxDeg}}$ is just smaller than $\tau$. We denote that cost as $\min C_{bj}^s$.

Let $C_{bj}^s$ be the external testing cost of $b_j$ with any skip factor $s$. We first show that $C_{bj}^s = \min C_{bj}^s$ when $s = |WCUR| - 1$. That is, we show $C_{bj}^s$ is instance optimal. If that is proved, then it is straightforward to deduce the total cost of all $b_j \notin R$ as minimum and thereby proved the lemma.

As mentioned above, if $C_{bj}^s = \min C_{bj}^s$, it implies that GMCV stops processing $b_j$ once its degree upper bound $b_{j,\text{maxDeg}}$ gets refined so that it is smaller than $\tau$, which is proved in Theorem 1.

## 5 Cost Model

Although SOE is proven to be instance optimal (i.e., for any given problem instance, it incurs at most a constant factor of tests of the optimal solution), it is not applicable to the context with group testing. In SOE, minimizing the number of tests is equivalent to minimizing the total external testing cost because the external cost of a 2-vertex test function is a constant. However, the overall external cost of a group test function depends not only on the number of tests invoked, but also on the input size to each test as well as the implementation of the group test.

In this section, we provide cost models to capture the total external testing costs of GMCV (Section 5.1) and SOE (Section 5.2) and compare their external costs based on different group test cost functions (Section 5.3). For every
black vertex \(b_i\), we assume that its degree \(d_i \neq 0\) and \(d_i \neq |W|\), as it is trivial to deal with these two cases.

### 5.1 External Testing Cost of GMCV

In an execution of GMCV, a particular black vertex \(b_i \in R\) is associated with a series of expanding-and-identifying processes that may span across multiple iterations. Initially, a test \(Q(b_i, W_j^1)\) is carried out. If \(Q(b_i, W_j^1) = 0\), another group test \(Q(b_i, W_j^2)\) is carried out. The expanding phase \(Q(b_i, W_j^1) = 0\), \(Q(b_i, W_j^2) = 0, \ldots\), continues until the \(s\)th test in which \(Q(b_i, W_j^s) = 1\) (while all the previous tests return 0), where \(s\) is called the turning point in the process. After that, the identifying phase starts: \(Q(b_i, W_j^{s+1}), \ldots, Q(b_i, W_j^{s+j-1})\), i.e., recursively drill into the set \(W_j^s\) to locate the solid vertex.

**Lemma 5.** Let \(C_i^j\) be the external testing cost of the \(j\)th expanding-and-identifying process of \(b_i\) and \(s\) be the turning point, then \(C_i^j = 2 \sum_{j=1}^{s-1} \beta(2^{-j}) + \beta(2^{-s})\).

**Proof.** Note that the size of the vertex set \(W_j^j\) has the following property:

\[
|W_j^j| = \begin{cases} 
2^{j-1}, & 1 \leq j \leq s \\
2^{2s-j-1}, & s < j \leq 2s-1.
\end{cases}
\]

As \(C_i^j\) denotes the external testing cost of the \(j\)th expanding-and-identifying process of \(b_i\), then \(C_i^j = \sum_{j=1}^{2s-1} \beta(|W_j^j|) = 2 \sum_{j=1}^{s-1} \beta(2^{-j}) + \beta(2^{-s})\).

**Lemma 6.** For \(b_i \in R\), the total external testing cost \(Cost(b_i)\) associated with \(b_i\) is

\[
Cost(b_i) = d_i \cdot C_i^j,
\]

where \(d_i\) is the degree of \(b_i\), and the value of turning point \(s\) in \(C_i^j\) is set as \(\lfloor \log \frac{|W_j^j|}{\alpha} \rfloor + 1\) (\(\lfloor x \rfloor\) denotes the randomized rounding [24] of \(x\)).

**Proof.** Lemma 5 gives the external testing cost of any expanding-and-identifying process. Since in GMCV, every black vertex \(b_i \in R\) is completed, i.e., it has the exact degree \(d_i\) and each expanding-and-identifying process locates one solid vertex, the cost of \(b_i \in R\) is thus \(d_i \cdot C_i^j\). The value of \(s\) in \(C_i^j\) is derived as follows:

An expanding-and-identifying process reveals 1 solid vertex plus at least \(\sum_{j=1}^{s-1} 2^{-j} = 2^{s-1} - 1\) empty vertices, a total of at least \(2^{s-1}\) vertices. Let \(\omega = 2^{s-1}\). Since the GMCV algorithm randomly picks white vertices to carry out group testing on \(b_i\), \(\omega\) can be approximated as \(|W|/d_i\). So, we have \(s - 1 = \lfloor \log \frac{|W|}{\alpha} \rfloor\), i.e., \(s = \lfloor \log \frac{|W|}{\alpha} \rfloor + 1\). We use randomized rounding here because \(s\) is an integer. □

Next, we derive \(Cost(b_i)\), the external testing cost associated with a vertex \(b_i \notin R\). Before that, we define \(A(t)\) be the accumulated external testing cost to identify \(t\) empty vertices through a series of group tests whose results are all zero (i.e., the external testing costs spent on the doubling strategy during the expanding phase). It is thus trivial to see that \(A(t) = \sum_{j=0}^{t} \beta(2^j)\).

**Lemma 7.** For \(b_i \notin R\), the external testing cost \(Cost(b_i)\) associated with \(b_i\) is

\[
Cost(b_i) = \lambda_j \cdot C_i^j + A(\theta - \lambda_j \cdot (2^s - 1)),
\]

where \(\theta = |W| - \tau + 1, \lambda_j = \lfloor \frac{\theta}{2^s-1} \rfloor, s = \lfloor \log \frac{|W|}{\alpha} \rfloor + 1\), and \(d_j\) is the degree of \(b_j\).

**Proof.** In Theorem 1, we show that a black vertex \(b_j \notin R\) does not need any further testing in GMCV, once its degree upper bound is smaller than \(\tau\). Meaning that \(b_j\) needs to detect \(|W| - (\tau - 1)\) empty vertices. Let \(\theta = |W| - (\tau - 1)\). Next, the analysis is redirected to analyze the external testing cost of detecting \(\theta\) empty vertices for \(b_j \notin R\).

As mentioned, an expanding-and-identifying process discovers 1 solid vertex plus at least \(2^{s-1} - 1\) empty vertices, where \(s = \lfloor \log \frac{|W|}{\alpha} \rfloor + 1\). Thus, to detect \(\theta\) empty vertices, it requires \(\lfloor \frac{\theta}{2^s-1} \rfloor = \lfloor \frac{\theta}{\lambda_j} \rfloor\) (denoted as \(\lambda\)) expanding-and-identifying processes.

For the remaining \(\theta - \lambda \cdot (2^s - 1)\) empty vertices, it requires a follow-up expanding phrase, which costs \(A(\theta - \lambda \cdot (2^s - 1))\).

Summing up the external testing cost gives the result, which completes the proof. □

**Theorem 2.** The external testing cost of GMCV is

\[
Cost_{GMCV} = \sum_{b_i \in R} Cost(b_i) + \sum_{b_j \notin R} Cost(b_j),
\]

where \(Cost(b_i)\) and \(Cost(b_j)\) are defined in Lemmas 6 and 7, respectively.

### 5.2 External Testing Cost of SOE

According to [29], [28], the number of tests \(X_{SOE}\) consumed by SOE for a hidden partite graph with \(|B|\) black vertices and \(|W|\) white vertices is...
\[ \mathcal{N}_{SOE} = k \cdot |W| + \sum_{i=k+1}^{|B|} \frac{(|W| - \tau + 1)(|W| + 1)}{l_i \cdot |W| + 1} \]

\[ = k \cdot |W| + \sum_{i=k+1}^{|B|} \frac{\theta(|W| + 1)}{|W| - d_i + 1}, \]

where \( l_i = 1 - \frac{d_i}{|W|} \).

Since each 2-vertex test has the cost of \( \beta(1) \), the external testing cost of SOE is

\[ \text{Cost}_{SOE} = \mathcal{N}_{SOE} \cdot \beta(1). \] (3)

### 5.3 Cost Comparison

We compare the external testing costs of GMCV and SOE based on the cost models established in (2) and (3). Following [29], [28], we assume the degrees of the bipartite graph follow power-law distribution such that for each \( b \in B \), its degree equals \( d \) (between 0 and \( |W| \)) has the probability:

\[ Pr(d) = \frac{1/(d + 1)^\gamma}{\sum_{d=0}^{|W|} 1/(i + 1)^\gamma}, \] (4)

where \( \gamma \) is the skewness factor to control the sparseness of a graph \( \gamma > 0 \). The smaller the \( \gamma \) is, the denser the graph is.

We consider four group testing implementations:

1. **Const**, where \( \beta(|W|) = \beta(1) \).
2. **Log**, where \( \beta(|W|) = \log |W| \cdot \beta(1) \).
3. **Sqrt**, where \( \beta(|W|) = \sqrt{|W|} \cdot \beta(1) \).
4. **Linear**, where \( \beta(|W|) = |W| \cdot \beta(1) \).

The **Const** implementation is to simulate the group test implementation in the biological domain, in which both the monetary cost and the running time of an experiment is a constant [17]. The **Log** and the **Sqrt** implementations are to simulate the group test implementations in the graph pattern matching and distance join applications, where the external cost (running time) is sublinear to the input size. Applications for the **Linear** group test implementation are not clear; however, we include it in our study to show that GMCV should not be misused in applications where the external cost of a group test is (super) linear to its input size.

Fig. 5 plots the external testing costs of GMCV and SOE \((k = 10)\) based on (2) and (3), on hidden partite graphs of varying sizes \(|B| = |W|\) and different sparseness \( \gamma \). It can be seen that GMCV outperforms SOE in almost all graph sizes and graph sparseness, except when the graphs are unusually dense \( (\gamma \text{ is close to } 0) \) or when GMCV is deliberately misused on applications where the external cost of a group testing is (super) linear to the size of the input. In those cases, we found GMCV and SOE have comparable performance.

### 6 Experiments

In this section, we evaluate GMCV on both real-life data sets and synthetic data sets.

**PPI.** \(^4\) It consists of the interactions between Yeast proteins, where \( B \) and \( W \) represent all the proteins. Particulariy, a protein \( b \in B \) connects with \( w \in W \) if they can interact with each other.

**Germany.** \(^5\) It is a real road network from Germany. In our problem setting, \( B \) and \( W \) contain all the nodes. A vertex \( b \in B \) and a vertex \( w \in W \) has an edge if their distance (in terms of the shortest path distance) is less than a predefined threshold, which is set to 10 km by default.

**Actor-W.** \(^6\) It is an actor collaboration network data based on IMDB (http://www.imdb.com). In which, \( B \) and \( W \)

---

3. Normally, \( \gamma \) is larger than 2.0 in real graphs [1], [2].
5. www.maproom.psu.edu/dcw.
include all the actors. In particular, two actors $b$ and $w$ have an edge if they have coappeared in at least one movie.

Actor-$D$, available from [29], [28]. It is derived from the actor collaboration social network data by extracting 10,000 actors that have the largest number of collaborators, i.e., $B$ and $W$. Two actors $b$ and $w$ have an edge if they have a two-hop relationship, i.e., either they appeared in at least one common movie, or they have a common collaborator.

Table 4 summarizes the properties of the four real data sets above. Actor-$D$ is unusually dense—in a hidden graph with only 10,000 black and 10,000 white vertices, a black vertex connects to more than 7,000 white vertices on average. In fact, Actor-$D$ does not follow power-law distribution as its $\gamma < 0$.

Synthetic Data. We follow [29], [28] to generate graphs of different sizes and sparseness. By default, $|B| = |W| = 5,000$.

Following [29], [28], we simulate the implementation of a (group) test. We use the four group testing functions Const, Log, Sqrt, and Linear mentioned in Section 5.3. For example, we regard the external cost of a group test with
an input of four vertices is 2, if the $\text{Sqrt}$ group test function is used. The experimental results are reported in terms of external testing cost.

6.1 Experimental Results on Real Data Sets

Fig. 6 (the bigger graph) shows the external testing costs of GMCV (based on different group testing cost functions) and SOE of different $k$ values, on the PPI data set. It is clear that GMCV outperforms SOE significantly, except when the inappropriate Linear group testing is deliberately used. Specifically, the costs of GMCV are 36 times ($\text{Const}$), 10 times ($\text{Log}$), and 7 times ($\text{Sqrt}$) less than SOE, respectively. Since their costs differ so much and we cannot see the effect of $k$ when putting them together in one graph, we plot their individual costs as well (smaller graphs). We can observe that all methods scale well with the value of $k$.

The experimental results on Germany and Actor-W data sets are shown in Figs. 7 and 8. We can also observe that GMCV outperforms SOE significantly, again except when the improper Linear group test function is deliberately used.

Fig. 9 shows the external testing costs of GMCV and SOE on Actor-D. We can see that even on such an unusually dense data set, SOE and GMCV have comparable performance. This is because GMCV uses the doubling strategy to adaptively determine the group size based on the outcome of the previous testing, i.e., double the group size if the previous test result is 0 and halve the group size otherwise. On dense graphs, however, a group testing has a high chance to return 1. Therefore, GMCV seldom employs the doubling strategy, which makes GMCV behave like SOE, but with a little overhead.

6.2 Experimental Results on Synthetic Data Sets

6.2.1 Sparseness

Fig. 10 shows the external testing costs of GMCV and SOE running on synthetic graphs of different sparseness. The skewness factor $\gamma$ ranges from 0.1 (average degree is 2,389) to 4.0 (average degree is 0.108). We can see that GMCV outperforms SOE from sparse to dense graphs, except when the improper Linear group test function is deliberately used. SOE is comparable with GMCV only when the graph is extremely dense ($\gamma = 0.1$).

6.2.2 Scalability

In this experiment, we evaluate the scalability of GMCV on synthetical graphs of different sizes (from 5,000 black vertices and 5,000 white vertices to 500,000 black vertices and 500,000 white vertices). The graphs here are generated using $\gamma = 2.0$, which is found in many real-life graph data [1], [2]. Fig. 11 shows the external testing costs of GMCV running on synthetic graphs of different sizes. We can see that GMCV scales well on graphs of different sizes.

7 Conclusions

This paper studies the $k$MCV problem on hidden bipartite graphs in the context of group testing. Group testing is a common testing model in hidden graph literature. Instead of testing the presence of edge between only two vertices (which is called the 2-vertex testing model), a group test takes as input a group of vertices and returns whether there is any edge among them. If group testing is used properly, a single group test can reveal the same information as multiple 2-vertex tests. Therefore, if the external cost of a group test is constant to or sublinear of the input size, the external cost of solving an $k$MCV problem can be significantly reduced. To that end, an algorithm that is based on group testing, called, GMCV, is developed. GMCV adaptively determines the size of the vertices to be input to each group test based on the data characteristics. Our cost analysis as well as experimental results show that GMCV outperforms SOE, a 2-vertex testing-based $k$MCV algorithm, except in some extreme cases (e.g., when the linear implementation of group testing is deliberately used or the graphs are unusually dense). In those cases, GMCV still has comparable performance with SOE, making GMCV a robust and more effective choice than SOE in the usual settings.
REFERENCES


Jianguo Wang received the bachelor’s degree from Zhengzhou University, China, in 2009 and the Mphil degree in computer science from The Hong Kong Polytechnic University in 2012. He is currently a PhD student at the University of California, San Diego. His research interests include data management system and new computing hardware.

Eric Lo received the PhD degree from ETH Zurich in 2007. He is currently an assistant professor in the Department of Computing, Hong Kong Polytechnic University. His research interests include query processing, query optimization, and large-scale data analysis.

Man Lung Yiu received the bachelor’s degree in computer engineering and the PhD degree in computer science from the University of Hong Kong in 2002 and 2006, respectively. Prior to his current post, he was with Aalborg University for three years starting in the Fall of 2006. He is currently an assistant professor in the Department of Computing, Hong Kong Polytechnic University. His research focuses on the management of complex data, in particular query processing topics on spatiotemporal data and multidimensional data.

*For more information on this or any other computing topic, please visit our Digital Library at [www.computer.org/publications/dlib](http://www.computer.org/publications/dlib).*