Nonmeasurable Sets

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In probability theory, we define a sample space X and a probability measure μ on some of the subsets of X. If X is countable, there is no harm in defining μ on all of $\mathcal{P}(X)$; but if X is, say, $[0, 1)$, then we can run into logical problems unless we restrict μ. In this essay, we demonstrate a nonmeasurable subset of $X = [0, 1)$ to show why this really is a concern.

Define μ to be the uniform measure on X with $\mu(X) = 1$. Define the equivalence relation $\sim$ on X by $x \sim y$ iff $x - y \in \mathbb{Q}$. Then we can partition X into the equivalence classes $\{[x] \mid x \in X\}$. Use the axiom of choice to define a choice function $c : \{[x] \mid x \in X\} \to X$ such that $\forall x \in X\ c([x]) \in [x]$. Define $S = \operatorname{im} c$ and for each $q \in \mathbb{Q}$, $S_q = \{s + q \mod 1 \mid s \in S\}$.

Claim 1. *X is the disjoint union $\cup_{q \in \mathbb{Q} \cap X} S_q$.*

Proof. (disjointness) Let $p, q \in \mathbb{Q} \cap X, p < q, s, s' \in S$, and suppose $s + p \mod 1 = s' + q \mod 1 \in S_p \cap S_q$. Then

$$\begin{aligned} &\exists z \in \mathbb{Z}\ s - s' = q - p - z \\ \Rightarrow &[s] = [s'] \\ \Rightarrow &s = s' \\ \Rightarrow &q - p = z, \end{aligned}$$

which contradicts that $0 < q - p < 1$.

(cover) Let $x \in X$. Then

$$x = c([x]) + (x - c([x])) \mod 1 \in S_{x - c([x])}.$$

□

Claim 2. *If S is measurable, then $\forall q \in \mathbb{Q}\ \mu(S_q) = \mu(S)$.*

Proof. Wlog suppose $q \in X$. Set

$$\begin{aligned} \alpha &= \{s + q \mid s \in S, s + q \leq 1\} \\ \beta &= \{s + q - 1 \mid s \in S, s + q > 1\} \\ \gamma &= \{s \in S \mid s \leq 1 - q\} \\ \delta &= \{s \in S \mid s > 1 - q\}. \end{aligned}$$

Then S_q is the disjoint union $\alpha \cup \beta$, S is the disjoint union $\gamma \cup \delta$, and $\alpha = \gamma + q, \beta = \delta + q - 1$. So by translation invariance (here we make use of the fact that μ is uniform),

$$\begin{aligned}\mu(S_q) &= \mu(\alpha) + \mu(\beta) \\ &= \mu(\gamma) + \mu(\delta) = \mu(S).\end{aligned}$$

□

Claim 3. *S is not measurable.*

Proof. Otherwise

$$\begin{aligned}1 = \mu(X) &= \sum_{q \in \mathbb{Q} \cap X} \mu(S_q) = \sum_{q \in \mathbb{Q} \cap X} \mu(S) \quad \text{by claim 2} \\ &= \begin{cases} 0 & \text{if } \mu(S) = 0 \\ \infty & \text{if } \mu(S) > 0 \end{cases}.\end{aligned}$$

□