Extension of Minimax to Infinite Matrices

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Abstract

Von Neumann’s minimax theorem is typically applied to a finite payoff matrix $A \in \mathbb{R}^{m \times n}$. Here we show that

(i) if $m, n$ are both infinite, then the conclusion of the minimax theorem may fail, and

(ii) if at least one of $m, n$ is finite and the entries of $A$ are bounded, then the conclusion of the minimax theorem holds.

1 Definitions

Consider a 2-player zero-sum game $G$ between players row $R$ and column $C$ where $R$ chooses an integer in $m$ and $C$ chooses an integer in $n$. The players choose simultaneously, or equivalently, sequentially but secretly. Then $C$ pays $A_{i,j}$ to $R$, where $A \in \mathbb{R}^{m \times n}$ is the payoff matrix. $R$ wants to maximize the payoff while $C$ wants to minimize the payoff.

For a countable set $\alpha$, let

$$D_{\alpha} = \{p \in \mathbb{R}^n \mid \forall i \in \alpha \; p_i \geq 0 \land \sum_{i \in \alpha} p_i = 1\}$$

denote the set of distributions on $\alpha$.

A pure strategy $i \in m$ ($j \in n$) for $R$ ($C$) is to choose a row $i$ (column $j$) with probability 1. A mixed strategy $p \in D_m$ ($q \in D_n$) for $R$ ($C$) is to choose each row $i \in m$ (column $j \in n$) with probability $p_i$ ($q_j$). If $R, C$ use mixed strategies $p, q$, then the expected payoff is $p^T A q = \sum_{i,j} p_i q_j A_{i,j}$.

2 Basic results

When developing a strategy, $R$ may consider what would happen if he were to give $C$ the following advantage: $R$ chooses first, reveals his choice to $C$, and then lets $C$ choose. If $i$ is a best pure strategy in this game $G^*$ and $R$ uses $i$ in the original game $G$, then $R$ can do no worse in $G^*$, and he could do exactly as bad. In other words, using $i$ maximizes the minimum possible payoff to $R$, and is therefore a very conservative strategy. $C$ could use the same strategy, minimizing the maximum possible payoff in the game $G^*$ where $C$ must move first.

This leads us naturally to compare $\sup_{i \in m} \inf_{j \in n} A_{i,j}$ with $\inf_{j \in n} \sup_{i \in m} A_{i,j}$. 
Lemma 1. Let $X, Y$ be nonempty sets and $f : X \times Y \to \mathbb{R}$. Then

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y),$$

even if one or both expressions are infinite.

Proof. Let $\epsilon > 0$ and choose $x^* \in X, y_* \in Y$ such that

$$\inf_{y \in Y} f(x^*, y) \geq \sup_{x \in X} \inf_{y \in Y} f(x, y) - \epsilon$$

$$\sup_{x \in X} f(x, y_*) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y) + \epsilon.$$

Then

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) - \epsilon \leq \inf_{y \in Y} f(x^*, y) \leq f(x^*, y_*) \leq \sup_{x \in X} f(x, y_*) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y) + \epsilon.$$

Since $\epsilon$ was arbitrary, the inequality follows.

This makes sense since in $G^*$, $R$ has a disadvantage compared to his position in $G_*$. 

Corollary 2. For $A \in \mathbb{R}^{m \times n}$ where $m, n \leq \omega$,

$$\sup_{i \in m} \inf_{j \in n} A_{i,j} \leq \inf_{j \in n} \sup_{i \in m} A_{i,j}.$$ 

The inequality above could be strict. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then the lhs is 0 while the rhs is 1.

We may abstract a bit and consider a choice for $R$ to be that he fixes a distribution $p \in D_m$. We again can consider what happens to the expected payoff if one player moves before the other, revealing his choice.

2.1 Defining the expected payoff

But there is a problem. We need to define the expected payoff. As long as $m, n$ are finite, there is no problem, but if either is infinite, then $p^t A q = \sum_{i,j} p_i q_j A_{i,j}$ could depend on the order in which the terms are added.

One way to resolve this to demand that $A$ be nonnegative. Then the summation is either absolutely convergent or properly divergent to $\infty$. Either way, the sum does not depend on the order. Another way to resolve the problem is to demand that $A$ be bounded.

The first solution is asymmetric since the expected payoff is then between 0 and $\infty$, which seems to favor $R$. Indeed, suppose $A$ is nonnegative and unbounded. If column $j$ is unbounded, say $\forall k \in \omega$ $A_{i_k,j} \geq k$ with $i_0 < i_1 < i_2 < \cdots$, and $C$ chooses $j$ with positive probability, then $R$ can use the strategy $p_{i_k} = \frac{6}{2^k - 1}$ to get an infinite expected payoff. In fact, $R$ can do this obliviously: $R$ does not need to know which unbounded column $C$ chooses with positive probability, only that some such column
exists. So it would be a bad idea for $C$ to choose any unbounded column with positive probability. If every column is unbounded, then the problem is uninteresting since $R$ can get an infinite expected payoff.

So now suppose that $C$ restricts himself to bounded columns. The same argument as before shows that he must restrict his probabilities so that the product of $q_j$ and the supremum of column $j$ is bounded, as a function of $j$.

To avoid all of these difficulties and to restore symmetry between players $R, C$, we will simply require that $A$ be bounded. We will soon see that the actual bounds do not matter so that we can choose bounds of $0, 1$ wlog.

2.2 Translation and scaling

Let $A \in \mathbb{R}^{m \times n}$ where $m, n \leq \omega$ and let $c \in \mathbb{R}$. Let $A + c = (A_{i,j} + c)$. Then $\forall p \in D_m, q \in D_n$

$$p^T (A + c) q = p^T A q + c,$$

and so the game is essentially the same, but the expected payoff is translated by $c$. If $c > 0$, then

$$p^T (cA) q = c(p^T A q).$$

If we scale by $-1$, then maximization and minimization are interchanged and we get

$$\sup_{p \in D_m} \inf_{q \in D_n} p^T (-A) q = - \inf_{q \in D_n} \sup_{p \in D_m} p^T A q. \quad (1)$$

By using translation and scaling, it is easy to translate results about matrices with entries coming from $[0, 1]$ into results for arbitrary bounded matrices. (In fact, we do not need scaling by $-1$ to conclude this, but it will come in handy later since it lets us transpose the matrix.)

2.3 Von Neumann’s theorem

Corollary 3. For $A \in \mathbb{R}^{m \times n}$ where $m, n \leq \omega$ and $A$ is bounded,

$$\sup_{p \in D_m} \inf_{q \in D_n} p^T A q \leq \inf_{q \in D_n} \sup_{p \in D_m} p^T A q.$$

Proof. Define $f : D_m \times D_n \to \mathbb{R}$ by $f(p, q) = p^T A q$, which is well-defined since $A$ is bounded, and apply lemma 1. \qed

Von Neumann’s minimax theorem, which we do not prove here, reverses the above inequality in the case that $m, n$ are finite, giving us the somewhat unexpected result that $R$ has the same advantage using a mixed strategy in both games $G^*$ and $G_*$.

Theorem 4 (von Neumann). For $A \in \mathbb{R}^{m \times n}$ where $m, n$ are finite,

$$\sup_{p \in D_m} \inf_{q \in D_n} p^T A q = \inf_{q \in D_n} \sup_{p \in D_m} p^T A q. \quad (2)$$
2.4 Counterexample for infinite matrices

We now show that (2) may fail when both $m, n$ are infinite. Define

$$A_{i,j} = \begin{cases} 
1 & \text{if } i \geq j \\
0 & \text{else}
\end{cases}.$$

I.e. the lower triangle is all 1’s while the upper triangle is all 0’s.

Claim 5.

$$\sup_{p \in D_m} \inf_{q \in D_n} p^T A q = 0 \quad (3)$$
$$\inf_{q \in D_n} \sup_{p \in D_m} p^T A q = 1. \quad (4)$$

Proof. We first show (3). Let $\epsilon > 0$ and let $p \in D_m$. $\exists k \in \omega \sum_{i=k}^{\infty} p_i < \epsilon$. Set $q = e_k$, the standard basis vector with a 1 in position $k$ and 0’s elsewhere. Then

$$p^T A q = \sum_{i,j} p_i q_j A_{i,j}$$
$$= \sum_i p_i A_{i,k}$$
$$= \sum_{i=k}^{\infty} p_i A_{i,k}$$
$$= \sum_{i=k}^{\infty} p_i < \epsilon,$$

which shows (3). The proof of (4) is analogous. \(\square\)

3 Main result

In light of the previous section, it is surprising that (2) does hold when one of $m, n$ is finite and the other is infinite.

The proof of this will use several techniques. At the heart of the argument, we will reduce the problem to the finite case, but getting there will be tricky.

Many times, we will use arguments of the form, “real numbers $x, y$ satisfy $\forall \epsilon > 0 \ x \leq y + \epsilon$, and therefore $x \leq y$.” Obviously the conclusion holds if we change the hypothesis to “$\forall \epsilon > 0 \ x \leq y + \epsilon$” where $\epsilon$ is an absolute constant not depending on $\epsilon$. Now let us complicate this. Fix $x, y \in \mathbb{R}$. Suppose we can show that $\exists c_2, \ldots, c_k \in \mathbb{R} \ \forall \epsilon > 0 \ \exists a_1, \ldots, a_k \in \mathbb{R}$ with $x = a_1, y = a_k$ such that

$$a_1 \leq a_2 + c_2 \epsilon$$
$$a_2 \leq a_3 + c_3 \epsilon$$
$$\vdots$$
$$a_{k-1} \leq a_k + c_k \epsilon.$$
Then we could conclude that \( x \leq y \). To see this, note that \( x \leq y + \sum_{i=2}^{k} c_i \varepsilon \).

The advantage of this last form of the argument is that the intermediate values \( a_i \) may depend on \( \varepsilon \) as long as the \( c_i \) do not.

With this in mind, once we have an \( \varepsilon \) in our context, we define the notions of almost inequality and almost equality. Say that \( x \) is almost less than or equal to \( y \) (\( x \preceq y \)) if there is an absolute constant \( c \), not depending on \( \varepsilon \) (even though \( x, y \) may depend on epsilon), such that \( x \leq y + c \varepsilon \). Also \( x \) is almost equal to \( y \) (\( x \sim y \)) if \( x \preceq y \land y \preceq x \).

Notice that \( \preceq, \sim \) are both reflexive and transitive and that \( \sim \) is symmetric. Also if \( x, y \) do not depend on \( \varepsilon \) and \( x \preceq y \), then \( x \leq y \). This notational trick will simplify life a great deal in the following proof.

**Theorem 6.** For \( A \in \mathbb{R}^{m \times n} \) where \( m, n \leq \omega \), at least 1 of \( m, n \) is finite, and \( A \) is bounded, \[
\sup_{p \in D_n} \inf_{q \in D_n} p^T A q = \inf_{q \in D_n} \sup_{p \in D_n} p^T A q.
\]

**Proof.** If both \( m, n \) are finite, this is just theorem 4. If \( m \) is finite and \( n \) is infinite, we can reduce to the case where \( m \) is infinite and \( n \) is finite by using (1) twice:

\[
\sup_{p} \inf_{q} p^T A q = - \inf_{q} \sup_{p} p^T (-A^T) q
\]
\[
= - \inf_{q} \sup_{p} p^T (-A^T) q
\]
\[
= \inf_{q} \sup_{p} p^T A q.
\]

So wlog, assume \( m = \omega \). By using translation and scaling, wlog assume that each \( A_{i,j} \in [0,1] \).

Half of the equality was shown in corollary 3. It remains to show \[
\sup_{p} \inf_{q} p^T A q \geq \inf_{q} \sup_{p} p^T A q.
\] (5)

Our next big step is to approximate each distribution in \( D_n \) by one of a finite number of distributions. Later this will let us exploit the continuity in \( p, q \) of the function \( p^T A q \). In particular, fix \( \varepsilon > 0 \) and choose \( Q = \{q^0, \ldots, q^N-1\} \subseteq D_n \) such that

- \( |Q| = N < \infty \)
- \( \forall q \in D_n \exists q' \in N \ |q - q'|_1 < \varepsilon \) where \( |q - q'|_1 = \sum_{i \leq n} |q_i - q'_i| \).

Any such \( Q \) will do, but if one needs a concrete example, take the integer lattice in \( \mathbb{R}^{n-1} \), scale it by \( \frac{1}{\sqrt{n-1}} \), then rotate it and translate it in \( \mathbb{R}^n \) to lie in the affine space defined by \( \sum x_i = 1 \). Finally let \( Q \) be the intersection of this with \( D_n \).
It is easy to see that for each \( q \in D_n \), if we choose \( l \in \mathbb{N} \) such that \( |q - q'|_1 < \epsilon \), then \( \forall p \in D_\omega \)

\[
|p^T Aq - p^T Aq'| \leq \sum_i |q_i - q'_i| \sum p_i A_{i,j} \\
\leq \sum_j |q_j - q'_j| \\
= |q - q'|_1 < \epsilon,
\]

and so

\[
 p^T Aq \sim p^T Aq'. \tag{6}
\]

For each strategy in \( Q \), we choose a counterstrategy. Specifically, \( \forall l \in \mathbb{N} \exists p^l \in D_\omega \)

\[
p^l^T Aq^l \geq \sup_p p^T Aq^l - \epsilon,
\]

and so

\[
p^l^T Aq^l \sim \sup_p p^T Aq^l. \tag{7}
\]

This finite number of counterstrategies makes use of only a finite number of rows w.h.p. and the remaining rows are used with only negligible probability. Specifically, \( \forall l \in \mathbb{N} \exists k \in \omega \sum_{i=k}^{\infty} p_i^l < \epsilon \), and so, taking the maximum \( k \) over all \( l \), we have

\[
\exists k \in \omega \forall l \in \mathbb{N} \sum_{i=k}^{\infty} p_i^l < \epsilon. \tag{8}
\]

Let \( B \in \mathbb{R}^{k \times n} \) be the first \( k \) rows of \( A \).

We now define 2 specific column strategies:

\[
\exists q_* \in D_n \sup_p p^T Aq_* \sim \inf_p p^T Aq \tag{9}
\]

\[
\exists q_* \in D_n \sup_p p^T Bq_* \sim \inf_p p^T Bq. \tag{10}
\]

Then

\[
\exists l_* \in \mathbb{N} \ |q_* - q^*|_1 < \epsilon \tag{11}
\]

\[
\exists l_* \in \mathbb{N} \ |q_* - q^*|_1 < \epsilon. \tag{12}
\]

Given \( p^l \in D_\omega \), we can define a similar distribution \( \tilde{p}^l \in D_k \) by

\[
\tilde{p}^l = \frac{p^l_{[k]}}{\sum_{i \in k} p_i^l}.
\]
Then we have that $\forall q \in D_n$, 
\[
|p^T Aq - \bar{p}^T Bq| \leq \left| \sum_{i \leq k} p_i q_j A_{i,j} - \frac{1}{\sum_{i' \in k} p_{i'}} q_j A_{i,j} \right| + \left| \sum_{i \geq k} p_i q_j A_{i,j} \right|
\]
\[
\leq \sum_{i \leq k} p_i \left( \frac{1}{\sum_{i' \in k} p_{i'}} - 1 \right) q_j + \epsilon
\]
\[
= 1 - \sum_{i \leq k} p_i + \epsilon < 2 \epsilon,
\]
and so
\[
p^T Aq \sim \bar{p}^T Bq.
\] (13) 

Finally we can combine all of this information. We have
\[
\sup_p p^T Aq_+ \sim \inf_q \sup_p p^T Aq
\]
by (9)
\[
\leq \sup_p p^T Aq_+ \sim \sup_p p^T Aq^l
\]
by (12) and (6)
\[
\sim p^T Aq^l
\]
by (7)
\[
\sim \bar{p}^T Bq^l
\]
by (13)
\[
\leq \sup_p p^T Bq^l \sim \sup_p p^T Bq
\]
by (12) and (6)
\[
\sim \inf_q \sup_p p^T Bq
\]
by (10)
\[
\leq \sup_p p^T Bq_+ \leq \sup_p p^T Aq_+;
\]
and since we have come full circle, we conclude that all of the intermediate expressions are almost equal. In particular, the $9^{th}$ and $10^{th}$ expressions are almost equal:
\[
\inf_q \sup_p p^T Bq \sim \sup_p p^T Bq_+.
\] (14) 

We finish the proof with another almost inequality chain, the middle of which uses theorem 4.
\[
\sup_p \inf_q p^T Aq \geq \sup_p \inf_q p^T Bq = \inf_q \sup_p p^T Bq
\]
by theorem 4
\[
\sim \sup_p p^T Bq_+
\]
by (14)
\[
\sim \sup_p p^T Bq^l_+ \geq \bar{p}^+ Bq^l_+ \sim p^+ Aq^l_+
\]
by (13)
\[
\sim \sup_p p^T Aq^l_+
\]
by (7)
\[
\sim \sup_p p^T Aq_+
\]
by (6)
\[
\sim \inf_q \sup_p p^T Aq
\]
by (9).

Since the first and last expressions do not depend on $\epsilon$, we conclude (5).