Theory of Jacob's Ladder

Chris Calabro

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Abstract

We give several characterizations for when a tree can be constructed from Jacob's ladder, such as iff it can be rooted such that each node has ≤ 1 non-leaf child, iff it can be written as a connected bipartite graph with no edge crossings.

1 Introduction

Jacob's ladder is an old toy, invented (almost certainly independently) all over the world. E.g. King Tut supposedly had one. Also, the mechanics form the basis for the Rubik's Magic puzzle. It is constructed from blocks numbered 1 through n (usually $n = 6$) and each of unit length in such a way that for $i \in \{1, \ldots, n-1\}$, the top of block i is always adjacent to the bottom of block $i+1$, or the bottom of block i is adjacent to the top of block $i+1$. It may be easier to think of a configuration as a directed graph with numbered edges.

2 Definitions

A graph is undirected unless it is said otherwise. If G is any kind of graph, let $U(G)$ be the undirected graph induced by G.

A graph G is *jacob* iff it is isomorphic to the undirected graph induced by a directed graph $([n+1], \{(a_1, b_1), \ldots, (a_n, b_n)\})$ such that each node touches an edge and $\forall i \in [n-1]$ $(a_i = a_{i+1} \vee b_i = b_{i+1})$. These are exactly the graphs constructible from Jacob's ladder.

A weakly connected bipartite graph (L, R, E) where L, R are each linearly ordered is *stacked* iff it has no edge crossings. An *edge crossing* is a tuple (a,b,c,d) with $\{a, b\}, \{c, d\} \in E$ and $a < c, d < b$. A graph G is *stackable* iff it is isomorphic to $U(G')$ for some stacked graph G'.

A graph G is *rootable* iff it is a tree and $\exists$ a root such that each node has ≤ 1 non-leaf child. Such a root is called an *appropriate* root.

3 Basic properties

Lemma 1. *If G is jacob, then G is a tree.*

Proof. We claim G is connected. Clearly, the nodes that touch edges are connected since $\forall i \in [n-1]$, (a_i, b_i) touches (a_{i+1}, b_{i+1}). Also every node touches an edge. So G is connected.

Next, a connected graph with n edges and $n+1$ nodes must be acyclic, else an edge could be removed and the graph would remain connected, contradicting that G contains a spanning tree. So G is a tree. □

Lemma 2. *If G is stackable, then G is a tree.*

Proof. G is connected by definition. Suppose indirectly that p is a cyclic path in G. Let x_1 be the least node on the left side of p. Let y_1 be a neighbor of x_1 in p. Let x_2 be the non-x_1 neighbor of y_1 in p. Then $x_2 > x_1$ since x_1 is least.

Suppose we have constructed $x_1 < \cdots < x_i$, $y_1 < \cdots < y_{i-1}$ such that $\forall j \in [i-1]$ $(x_j, y_j), (x_{j+1}, y_j) \in p$. Let y_i be the non-y_{i-1} neighbor of x_i. Then $y_i > y_{i-1}$, else $(x_{i-1}, y_{i-1}, x_i, y_i)$ is an edge crossing. Similarly, let x_{i+1} be the non-x_i neighbor or y_i. Then $x_{i+1} > x_i$, else $(x_{i+1}, y_i, x_i, y_{i-1})$ is an edge crossing.

So $x_1 < x_2 < \cdots$, implying that p is infinite, a contradiction. □

Lemma 3. *If G is rootable, then some leaf is an appropriate root.*

Proof. Let r be an appropriate root. If r is not a leaf, then since it has ≤ 1 non-leaf child and has > 1 child, it must have a leaf child x. Clearly x is an appropriate root. □

4 Equivalence

Lemma 4. *If G is jacob, then G is stackable.*

Proof. We will construct a sequence $G_1, \ldots, G_n$ of bipartite graphs such that

1. $G_i = (\{x_1, \ldots, x_{l_i}\}, \{y_1, \ldots, y_{r_i}\}, \{(a_1, b_1), \ldots, (a_i, b_i)\})$
2. $(x_{l_i}, y_{r_i}) = (a_i, b_i)$
3. $U(G_i)$ is an edge-induced subgraph of G
4. $x_1, \ldots, x_{l_i}, y_1, \ldots, y_{r_i}$ are distinct and $l_i + r_i = i + 1$
5. G_i is stacked (with node order $x_1 < \cdots < x_{l_i}, y_1 < \cdots < y_{r_i}$).

Set $x_1 = a_1$, $y_1 = b_1$, $l_1 = r_1 = 1$. Clearly the base case holds. Now suppose G_{i-1} has been constructed and we wish to construct G_i. Wlog suppose $a_i = a_{i-1}$. Set $l_i = l_{i-1}, r_i = r_{i-1} + 1, y_{r_i} = b_i$. Then $(x_{l_i}, y_{r_i}) = (x_{l_{i-1}}, b_i) = (a_{i-1}, b_i) = (a_i, b_i)$.

Since $U(G_{i-1})$ is an edge-induced subgraph of G and $\{a_i, b_i\}$ is an edge of G, $U(G_i)$ is an edge-induced subgraph of G.

Since G_{i-1} is weakly connected and $a_i = x_{l_{i-1}}$, G_i is weakly connected. G_i has i edges and $\leq l_i + r_i = l_{i-1} + r_{i-1} + 1 = i + 1$ nodes. Suppose indirectly that

G_i has $\leq i$ nodes. Then $U(G_i)$ is a cyclic subgraph of G, contradicting that G is a tree (lemma 1). So G_i has $i+1$ nodes.

G_{i-1} is bipartite and the new edge $(a_i, b_i) = (x_{l_i}, y_{r_i})$ goes from left to right, so G_i is bipartite. Finally, suppose indirectly that G_i has an edge crossing. Since G_{i-1} has no edge crossings, one of the edges in the edge crossing of G_i must be $(a_i, b_i) = (x_{l_i}, y_{r_i})$ and the other edge is (x_u, y_v). But then $u > l_i$ or $v > r_i$, a contradiction.

Having constructed the G_i, we see that $U(G_n) = G$ and so G is stackable. ☐

Lemma 5. *If G is stackable, then G is rootable.*

Proof. By lemma 2, G is a tree. We claim that the least node r on the left is an appropriate root.

First we claim that if x is a node in G with parent p, then p is less than the children of x. For suppose not. Since G is connected and p is the parent of x, there is a path from r to p not involving x. If some child y of x is less than p, then there must be an edge crossing involving edge (x, y) and an edge in the path from r to p, a contradiction.

So now suppose indirectly that some node x has non-leaf children $y_1 < y_2$. Let z be a child of y_1.

If $x = r$, then since r is least, $r < z$. But then (r, y_2, z, y_1) is an edge crossing. On the other hand, if $x \neq r$, then let p be the parent of x. Previously, we showed that $p < y_1 < y_2$. If $z < x$, then (z, y_1, x, p) is an edge crossing. If $z > x$, then (x, y_2, z, y_1) is an edge crossing. All cases lead to a contradiction, so each node x has ≤ 1 non-leaf child. ☐

Lemma 6. *If G is rootable, then G is jacob.*

Proof. First order the children of each node so that the non-leaf child, if there is one, comes last. The breadth-first ordering of the edges shows that G is jacob, provided we orient the edges in odd levels upwards and edges in even levels downwards (or vice versa). ☐

Theorem 7. *The following are equivalent.*

- *G is jacob*
- *G is stackable*
- *G is rootable*

5 Conclusions

Of the 3 definitions, jacob is closest to the physically constructed toy. But a certificate for whether a graph is jacob is linear in size, involving a permutation and orientation of the edges. So it is hard to directly find such a certificate, although verification is fast. Stackability is also fairly close to the physical construction but also has a linear-sized certificate. On the other hand, rootability has a log-sized certificate and so can be exhaustively checked in polynomial time.

The proof of lemma 6 shows how to construct a given graph with Jacob's ladder. But the equivalence theorem is most powerful when trying to quickly prove that a given graph is *not* constructible. Lemma 3 can be used to speed this up slightly.

It should be noted that the jacob graphs can be visualized easily: such a graph will always have a *spine*, a linear subgraph, with all other edges sticking out from the spine. The spine is a vertex cover. So, provided it is drawn nicely, one can spot a jacob graph immediately.

The number of isomorphism classes of graphs with $1, 2, 3, 4, 5, 6$ nodes is $1, 1, 1, 2, 4, 6$. They are all rootable, and therefore constructible by Jacob's ladder. The smallest non-rootable graph has size 7 and edges $\{1, 2\}, \{2, 3\}, \{1, 4\}, \{4, 5\}, \{1, 6\}, \{6, 7\}$. It is shaped like a 'Y'. To see that this is not rootable, it is sufficient to check that any single leaf is not an appropriate root, since by symmetry, all leaves will then be inappropriate, and, by lemma 3, all nodes will then be inappropriate. So this 'Y' cannot be constructed by Jacob's ladder.

It is curious that the Jacob's ladder toy is usually constructed with $n = 6$, since this is the first size at which a non-constructible graph appears. Perhaps $n = 6$ was chosen so that many graphs could be constructed, but not so many that the restricted form of such graphs would be apparent.