Infinite Dimensional Matrices

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Let $V = \mathbb{R}^\omega$ be the vector space of denumerable tuples of reals. Notice that V has dimension $2^{\aleph_0}$, and not $\aleph_0$. Let $l_2 = \{x \in V \mid \sum x_i^2 < \infty\}$ be the set of l_2 bounded vectors. We would like to characterize those matrices $A \in \mathbb{R}^{\omega\times\omega}$ which map $l_2 \to l_2$. Unfortunately, we will not get a complete characterization in this essay.

1 Nonmatching necessary and sufficient conditions

First let us observe an interesting, though perhaps not entirely relevant, convergence lemma.

Lemma 1. $\exists x \in l_2\ \sum \frac{x_i}{\sqrt{i}} = \infty$.

Proof. [1] Let $x_i = \frac{1}{\sqrt{i}\ln i}$. Now x_i^2, when viewed as a step function, is nonincreasing and so we may upperbound it by the continuous function $\frac{1}{x\ln^2 x}$. So

$$\sum_{i=2}^{\infty} x_i^2 \le \frac{1}{2} + \int_2^\infty \frac{1}{x\ln^2 x}dx,$$

which is easily seen to converge by using the substitution $u = \ln x$.

Next, $\frac{x_i}{\sqrt{i}}$, when viewed as a step function, is lower bounded by the continuous function $\frac{1}{x\lg x}$. So

$$\sum_{i=2}^{\infty} \frac{x_i}{\sqrt{i}} \ge \int_2^\infty \frac{1}{x\ln x}dx,$$

which is easily seen to properly diverge by using the same substitution $u = \ln x$. □

Let us suppose $A : l_2 \to l_2$. We claim that each row and column of A must be in l_2. For let a_j be the j^{th} column of A. $e_j \in l_2$ and so we must have $Ae_j = a_j \in l_2$. To see that each row is in l_2 will be much harder.

[1] proof simplified by Yi-Kai Liu

Theorem 2. $\forall y \in V - l_2 \; \exists x \in l_2 \; x \cdot y = \infty$.

Proof. Define recursively $A(k) = \min\{n \geq 0 \mid \sum_{j=A(k-1)}^{n-1} y_j^2 \geq 1\}$ for $k \geq 1$ and $A(0) = 1$. The idea is that we want to break up y into intervals such that the k^{th} interval, which is between $A(k-1)$ and $A(k) - 1$, has l_2 norm at least 1.

Define $y_k = (y_{A(k-1)}, \ldots, y_{A(k)-1})$ to be the k^{th} interval of y. Set $x_k = \frac{1}{k} \frac{y_k}{|y_k|}$ and x to be the concatenation of the x_k's. Then

$$|x|^2 = \sum_k |x_k|^2 = \sum_k \frac{1}{k^2} < \infty$$

and

$$x \cdot y = \sum_k x_k \cdot y_k = \sum_k \frac{1}{k} |y_k| \geq \sum_k \frac{1}{k} = \infty.$$

$\square$

So each row of A is in l_2. These conditions are not sufficient to ensure that $A : l_2 \to l_2$. For consider the example where A is diagonal with $A_{i,i} = i$. Let $x_i = \frac{1}{i}$. Then each row and column of A is in l_2, $x \in l_2$, and yet $Ax \notin l_2$.

This suggests that perhaps we need a *uniform* upper bound on the l_2 norms of the rows and columns of A. I don't know whether this assertion is correct. But let us at least give one sufficient condition for $A : l_2 \to l_2$. If the vector of l_2 norms of the rows of A is itself in l_2, then $A : l_2 \to l_2$.

2 Open problems

If $A, B : l_2 \to l_2$, does it follow that $AB : l_2 \to l_2$? This is less obvious than it may at first appear. One may think that if $x \in l_2$, then $(AB)x \in l_2$ since $Bx \in l_2$. But is matrix multiplication associative? Actually no. To see this, let x be the all 1's vector, A be a permutation matrix, and y contain the terms of the alternating harmonic series, i.e. $y_i = \frac{(-1)^i}{i}$. Then $(xA)y$ is some constant independent of A, while $x(Ay)$ could be anything at all, depending on A: it could be any real number, it could properly diverge, or it might not converge at all. Notice that A was actually a very nice matrix. Its rows and columns were uniformly l_2 bounded. Also y was very nice, being in l_2.

But it may be that for matrices mapping $l_2 \to l_2$, matrix multiplication is associative.

What are the eigenvalues of A? Could there be a continuum of them? Is the characteristic polynomial p a power series? Is is analytic? Is its radius of convergence infinite? If analytic, then p cannot have a continuum of roots within its interval of convergence, unless $p = 0$. To see this, suppose $p(x) = \sum \frac{p^{(n)}(a)}{n!}(x-a)^n$ where a is in the interior of the interval of roots. Then each $p^{(n)}(a) = 0$, and so p would be 0 everywhere.

On the other hand, p can have a continuum of roots. For let A be the matrix 0's everywhere except that it has all 1's on the first superdiagonal. Then every $a \in \mathbb{R}$ is an eigenvalue with eigenvector $(1, a, a^2, a^3, \dots)$.

Left multiplication by A is a linear map, but can all linear maps from $V \to V$ be represented by a matrix? No. To see this, note that the number of endomorphisms on V is $|V|^{\dim V} \geq 2^{2^{\aleph_0}} > 2^{\aleph_0} = |\{\text{matrices A}\}|$. This same reasoning carries through even if we restrict ourselves to the subspace space l_2.

Lemma 3. $\dim l_2 = 2^{\aleph_0}$.

Proof. We use a variant of the basic proof that $\dim V = 2^{\aleph_0}$. To upperbound $\dim l_2$, note that $|l_2| \leq |V| = |2^{\omega^2}| = |2^\omega| = 2^{\aleph_0}$.

To lowerbound $\dim l_2$, we construct an explicit set of $2^{\aleph_0}$ linearly independent vectors. We define $f : 2^\omega \to l_2$ as follows. Let $f(x)(0) = 0$ and for $2^{k-1} \leq i < 2^k$, let

$$f(x)(i) = \begin{cases} \frac{1}{i} & \text{if } i - 2^{k-1} = \sum_{j=0}^{k-1} 2^{x(j)} \\ 0 & \text{else} \end{cases}.$$

The idea here is that we divide ω into intervals. For $k = 1, 2, \dots$, the k^{th} interval has size 2^{k-1}. We encode $x : \omega \to 2$ into $y \in \mathbb{R}^\omega$ as follows: the k^{th} interval of y will encode the restriction of x to $k = \{0, \dots, k-1\}$ by writing $x \mid k$ out in binary as b and then putting a $\frac{1}{i}$ into the b^{th} position of the k^{th} interval. All other entries of y are 0. $f(x) \in l_2$ since its l_2 norm is upper bounded by the series $\sum \frac{1}{i^2} < \infty$.

Let $x_1, \dots, x_n \in 2^\omega$ be distinct. Then $\exists k-1 \in \omega$ such that $x_1 \mid k, \dots, x_n \mid k$ are distinct as well. In the k^{th} interval, $f(x_i)$ will have all 0's except for a single nonzero entry in position $b_i = \sum_{j=0}^{k-1} 2^{x_i(j)}$. But the b_i's are distinct, and so the $f(x_i)$ are independent. $\square$

So what kinds of linear maps can be represented by matrices?