Dual of a Vector Space

Chris Calabro
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1 What is a dual space?

Let $V$ be a vector space over a field $F$. Then the dual of $V$ is

$$V^* = \text{Hom}_F(V, F),$$

the set of $F$-linear homomorphisms from $V$ to $F$, and indeed this is a vector space over $F$.

2 $V, V^*$ isomorphic when $\dim_F V < \infty$

Claim 1. Suppose $\dim_F V < \infty$. Then $V \cong V^*$.

Proof. Let $B = \{b_1, \ldots, b_n\}$ be a basis for $V$. Define $\phi : V \to V^*$ by

$$\phi(\sum a_i b_i) = (b_i \mapsto a_i),$$

where the $a_i \in F$. In other words, $\phi(\sum a_i b_i)$ is the function $g : V \to F$ defined by $g(\sum c_i b_i) = \sum c_i a_i$, where the $c_i \in F$.

We claim that $\phi$ is an isomorphism. First let us show that $\phi$ is a homomorphism:

$$\phi(\sum a_i b_i + \sum a'_i b_i) = \phi(\sum (a_i + a'_i) b_i)$$
$$= (b_i \mapsto a_i + a'_i)$$
$$= (b_i \mapsto a_i) + (b_i \mapsto a'_i)$$
$$= \phi(\sum a_i b_i) + \phi(\sum a'_i b_i)$$

and

$$\phi(c \sum a_i b_i) = \phi(\sum ca_i b_i)$$
$$= (b_i \mapsto ca_i)$$
$$= c(b_i \mapsto a_i)$$
$$= c\phi(\sum a_i b_i).$$
Next let us look at the kernel of \( \phi \). Suppose \( \phi(\sum a_i b_i) = (b_i \mapsto 0) \). Then each \( a_i = 0 \) and hence \( \sum a_i b_i = 0 \). So \( \phi \) is injective.

Next we show that \( \phi \) is surjective. Let \( g \in \text{Hom}_F(V, F) \) and suppose \( g(b_i) = a_i \) for each \( i \in \{1, \ldots, n\} \). Then \( \phi(\sum a_i b_i) = g \). Notice carefully that it is here that we use the assumption that \( n < \infty \), since this allows the sum to be finite.

So indeed, \( \phi \) is an isomorphism. \( \square \)

3 \( V, V^* \) not necessarily isomorphic when \( \dim_F V = \infty \)

We will present a counterexample. We will use the fact that the cardinals are well ordered. In particular, let \( \omega \) be the least infinite cardinal.

Recall the cardinal arithmetic theorem: if \( \alpha \leq \beta \) are cardinals and \( \beta \geq \omega \), then \( \alpha + \beta = \alpha \cdot \beta = \beta \). Let \( F = 2 = \{0, 1\} \) be the field of size 2. Then we claim that \( V = 2^\omega \nsubseteq V^* \).

To see this, we first find the dimension of \( V \). Let \( v_i \in V \) be the vector with a 1 in the \( i \)th position and 0’s elsewhere. Then the \( v_i \) are linearly independent and so the dimension of \( V \) is infinite. Let \( B \) be a basis for \( V \).

\[
|V| = \left| \bigcup_{n \in \omega} \bigcup_{C \subseteq B, |C| = n} \left\{ \sum_{i=1}^{n} a_i c_i \mid a_i \in 2, c_i \in C \right\} \right|
\]

\[
\leq \sum_{n \in \omega} \sum_{C \subseteq B, |C| = n} 2^n
\]

\[
\leq \sum_{n \in \omega} |B|^n 2^n
\]

\[
= \sum_{n \in \omega} |B| \quad \text{by cardinal arithmetic and } |B| \geq \omega
\]

\[
= \omega \cdot |B|
\]

\[
= |B| \quad \text{by cardinal arithmetic and minimality of } \omega
\]

\[
\leq |V|.
\]

So \( |B| = |V| = 2^\omega \). Now \( |V^*| = |\text{Hom}_F(V, F)| = |F|^{\dim_F V} = 2^{2^\omega} > 2^\omega = |V| \), and so \( V \nsubseteq V^* \).