Convex Optimization Problem:

A. Convex Function Definition:

\[ f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), \forall \alpha + \beta = 1, \alpha, \beta \geq 0 \]

Convex Optimization Problem:

B. Convex Set Definition: \( \forall x, y \in C \)

We have \( \alpha x + \beta y \in C, \forall \alpha + \beta = 1, \alpha, \beta \geq 0 \)
Chapter 2 Convex Set

1. Set Convexity and Specification
   i. Convexity
   ii. Implicit vs. Explicit Enumeration

2. Convex Set Terms and Definitions
3. Separating Hyperplanes
4. Dual Cones

1. Set Convexity and Specification: Convexity

A set $S$ is convex if we have

\[ \alpha x + \beta y \in S, \forall \alpha + \beta = 1, \alpha, \beta \geq 0, \forall x, y \in S \]

Remark: The definition implies the following

1. Most used sets in the class
   1. Scalar set: $S \subset \mathbb{R}$
   2. Vector set: $S \subset \mathbb{R}^n$
   3. Matrix set: $S \subset \mathbb{R}^{n \times m}$
2. Set $S$ is convex if every two points in $S$ has the connected straight segment in the set.
3. For convex sets $S_1$ and $S_2$: $S_1 \cap S_2$ is also convex

   \[ \text{If } x \in S_1 \cap S_2 \Rightarrow x \in S_1, x \in S_2 \]
   \[ \text{Then } x, y \in S_1 \cap S_2 \Rightarrow \alpha x + \beta y \in S_1 \]

   \[ \text{If } x \in S_1 \cap S_2 \Rightarrow x \in S_1, y \in S_2 \]
   \[ \text{Then } \alpha x + \beta y \in S_1 \cap S_2, \alpha + \beta = 1, \alpha, \beta \geq 0 \]

1. Set Convexity and Specification: Convexity

Set Specification via Implicit or Explicit Enumeration

**Implicit Expression**

\[ S_I = \{ x | Ax \leq b, x \in \mathbb{R}^n \} \]

**Explicit Expression**

\[ S_E = \{ Ax | x \in \mathbb{R}_+^n \} \]

Implicit Expression: Constraints

Min $f_0(x)$

Subject to

\[ Ax \leq b, x \in \mathbb{R}^n \]

Explicit Expression: Enumeration

Min $f_0(Ax)$, $x \in \mathbb{R}^n_+$
1. Implicit vs Explicit Enumeration of Convex Set

**Implicit Expression**

\[ S_1 = \{ x | Ax \leq b \} \]

Example: \( \{ x | Ax \leq b \} \)

- \( x_1 + 2x_2 + 3x_3 \leq 4 \)
- \( 2x_1 - x_2 \leq 3 \)
- \( x_2 + x_3 \leq 5 \)
- \( x_3 \leq 10 \)

Remark: Simultaneous linear constraints imply AND,
Intersection of the constraints

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 10 \end{bmatrix} \]

Simultaneous linear inequalities form a convex set

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1. Implicit vs Explicit Enumeration of Convex Set

Example:

- \( S_1 = \{ x | Ax \leq b, x \in \mathbb{R}^n \} \)
- \( S_2 = \{ x | Ax \geq b, x \in \mathbb{R}^n \} \) convex? Yes
- \( S_3 = \{ x | Ax = b, x \in \mathbb{R}^n \} \) convex? Yes

If \( Ax \geq b \), then we can write

\[ -Ax \leq -b \]

Let \( A' = -A \), then we have \( A'x \leq b \).

If \( Ax = b \), then \( Ax \leq b \) & \(-Ax \leq b \). Let \( A' = \begin{bmatrix} A \\ -A \end{bmatrix} \), then \( A'x \leq b \).

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1. Specification of Convex Set: Implicit Expression

Example:

\[ S = \{ x \in \mathbb{R}^n | |p_x(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \} \] is convex

where \( p_x(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt \)

If \( x, y \in S \) (t, x, y \in \mathbb{R})

\[ \begin{align*}
  p_x(t) &= x_1 \cos t + \cdots + x_m \cos mt \\
  p_y(t) &= y_1 \cos t + \cdots + y_m \cos mt \\
  p_{x+y}(t) &= (x_1 + y_1) \cos t + \cdots + (x_m + y_m) \cos mt \\
  p_{x+y}(t) &= \theta p_{x+y}(t) \\
  p_{x+y}(t) &\leq 1
\end{align*} \]
1. Implicit vs Explicit enumeration of Convex Set

**Explicit Enumeration**

Statement: $S_5$ is convex if $C_5$ is convex.

Proof: Given $\left(\frac{z_1}{t_1}\right) \in S_5$, $\left(\frac{z_2}{t_2}\right) \in S_5$, let us set

$$z_3 = \alpha z_1 + \beta z_2, t_3 = \alpha t_1 + \beta t_2, \forall \alpha + \beta = 1, \alpha, \beta \geq 0$$

We have

$$\frac{z_3}{t_3} = \frac{\alpha z_1 + \beta z_2}{\alpha t_1 + \beta t_2} = \frac{\alpha t_1}{\alpha t_1 + \beta t_2} \frac{z_1}{t_1} + \frac{\beta t_2}{\alpha t_1 + \beta t_2} \frac{z_2}{t_2}$$

Let

$$\alpha' = \frac{\alpha t_1}{\alpha t_1 + \beta t_2}, \beta' = \frac{\beta t_2}{\alpha t_1 + \beta t_2}$$

(Note that $\alpha' + \beta' = 1, \alpha', \beta' \geq 0$),

we have

$$\frac{z_3}{t_3} = \frac{\alpha' z_1}{t_1} + \beta' \frac{z_2}{t_2} \in S_5$$

Therefore, by definition $S_5$ is convex.