Midterm Review for CSE 203B

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Based on slides by Fangchen Liu
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Logistics

- Released on course website:
  [http://cseweb.ucsd.edu/classes/wi21/cse203B-a/](http://cseweb.ucsd.edu/classes/wi21/cse203B-a/)
- Full 48 hours, submission on gradescope
- Released Tuesday 2/16 10:00 am PST, due Thursday 2/18 10:00 am PST
- 2 sections:
  - ~ 10 True/False (with explanation)
  - ~ 5 Derivations/simple proofs
  - At least one programming question
  - ~ 70% based on homework questions
- No questions will be answered on piazza (sorry!)
Overview

- Convex sets
- Convex separation
- Convex functions
- Conjugate function
- Lagrangian Dual
- Logistics and other recommended topics
Convex sets: definition

A set $S \subseteq \mathbb{R}^d$ is convex if the line segment between any two points in $C$ lies in $C$: for any $x_1, x_2 \in C$ and $0 \leq \theta \leq 1$, $\theta x_1 (1 - \theta) x_2 \in C$

Example: the polytope $\mathcal{K} = \{x | Ax \leq b\}$ for $x, b \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times n}$
Supporting Hyperplane Theorem

A supporting hyperplane to a set $C$ is defined with respect to a boundary point $x_0$:

$$\{x | a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$.

Supporting hyperplane theorem: If $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$. 
Separating Hyperplane Theorem

If $C$ and $D$ are nonempty disjoint convex sets, there exists $a \neq 0$ s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$

The hyperplane $\{x | a^T x = b\}$ separates $C$ and $D$. Strict separation requires additional assumptions (e.g. $C$ is closed, $D$ is a singleton).
Convex functions: definition

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\text{dom}f$ is a convex set and if for all $x, y \in \text{dom}f$ and $0 \leq \theta \leq 1$
  
  $$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$  
  
  Jensen’s inequality

- Concave functions: $-f$ is convex
Convex functions: definition

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- Concave functions: $-f$ is convex
Convex functions: first order condition

If $f$ is differentiable (dom$f$ is open, $\nabla f$ exists $\forall x \in$ dom$f$) then $f$ is convex iff dom$f$ is convex and for all $x, y \in$ dom $f$

$$f(y) \geq f(x) + \nabla f(x)^{T}(y - x)$$
Convex functions: second order condition

Suppose $f$ is twice-differentiable ($\text{dom} f$ is open and its Hessian exists $\forall x \in \text{dom} f$) then $f$ is convex iff $\text{dom} f$ is convex and for all $x, y \in \text{dom} f$:

$$\nabla^2 f \succeq 0 \quad \text{(positive semidefinite)}$$
Convex functions: establishing convexity

By definition

- Show by definition or first-order condition
- For twice-differentiable functions, show $\nabla^2 f \succeq 0$

By convexity-preserving operations

- Nonnegative weighted sum
- Composition with affine function
- Pointwise maximum and supremum
- Composition
- Minimization
- Perspective
Convex functions: establishing convexity

By definition

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By convexity-preserving operations

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- Perspective
Convex functions: relationship with convex sets

- A function is convex iff its epigraph is a convex set
- Consider a convex function $f$ and $x, y \in \text{dom} f$
  
  $$t \geq f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

- The hyperplane supports $\text{epi} f$ at $x, f(x)$, for any $y, t \in \text{epi} f$
  
  $$\implies \nabla f(x)^T (y - x) + f(x) - t \leq 0$$
Convex functions: examples

powers of absolute value

\[ f = |x|^p \] is convex on \( \mathbb{R}^+ \) and \( p > 1 \)

Pf: Note that the composition of a convex and convex-increasing function is convex. Prove \( |\cdot| \) is convex and \( x^p \) is convex and increasing.

TODO: Show log-convex function is convex (\( g(x) = \log(f(x)) \)), s.t. \( f \) convex. (first show for \( f \) twice-differentiable)
Convex functions: examples

quadratic form of inverse

\[ f : \mathbb{R}^n \times S^n \rightarrow \mathbb{R}, \quad f(x, Y) = x^T Y^{-1}x \] is convex on \( \mathbb{R}^n \times S^n_{++} \)

Pf: Show epigraph of \( f \) is a convex set. Express epigraph as an LMI and apply the definiteness conditions of the Schur Complement (appendix 5.5).
Conjugate function

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the conjugate function

$$f^*(x) = \sup_{x \in \text{dom} f} y^T x - f(x)$$

- $\text{dom} f^*$ consists of $y \in \text{dom} f$ such that $\sup_{y \in \text{dom} f} y^T x - f(x)$ is bounded.

- $f^*(x)$ is convex even if $f(x)$ is not convex.
Pointwise supremum

- If for each $y \in U$ $f(x, y)$ is convex in $x$, then
  
  $$g(x) = \sup_{x \in U} f(x, y)$$

  is convex in $x$.

- Example: $f^*(x) = \sup_{x \in \text{dom} f} y^T x - f(x)$

- Example: First order condition for convex functions
Duality

Primal problem

\[
\begin{align*}
\text{min } & \quad f_0(x) \\
\text{subject to } & \quad f_i(x) \leq 0 \\
& \quad h_i(x) = 0
\end{align*}
\]

Lagrange dual function \( g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \)

\[
g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)
\]

\[
= \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)
\]
Duality

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$$

$$= \inf_{x \in D} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

$g$ is concave, can be unbounded for some $-\lambda, \nu$.

Lower bound property
If $\lambda \geq 0$, then $g(\lambda, \nu) \leq p^*$.

proof: if $\bar{x}$ is feasible and $\lambda \geq 0$ then

$$f_0(\bar{x}) \geq L(\bar{x}, \lambda, \nu) \geq \inf_{x \in D} L(\bar{x}, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible $\bar{x}$ gives $p^* \geq g(\lambda, \nu)$. 
Duality

Lagrange dual function $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$

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minimizing over all feasible $\bar{x}$ gives $p^* \geq g(\lambda, \nu)$. 
Duality example: Primal and Dual of an LP

\[
\min_x c^T x \\
\text{s.t. } Ax \leq 0
\]

- The feasible set is the polytope \( K = \{x | Ax \leq b\} \)
- The Lagrange dual function of the primal problem is

\[
g(\lambda) = \inf_x (c^T x + \lambda A x) = \begin{cases} 
0 & A^T \lambda + c = 0, \lambda \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

- The dual problem is

\[
\max 0 \\
\text{s.t. } A^T \lambda + c = 0 \\
\lambda \geq 0
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\]

- The feasible set is the polytope \( K = \{ x \mid Ax \leq b \} \)
- The Lagrange dual function of the primal problem is

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0 & A^T \lambda + c = 0, \lambda \geq 0 \\
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\end{cases}
\]

- The dual problem is

\[
\begin{align*}
\max & \quad 0 \\
\text{s.t.} & \quad A^T \lambda + c = 0 \\
& \quad \lambda \geq 0
\end{align*}
\]

- Farkas lemma: \( Ax \leq 0, \quad c^T x < 0 \) where \( A \in \mathbb{R}^{m \times n}, \ c \in \mathbb{R}^n \) is satisfied for some \( x \) iff \( A\lambda = c \) s.t. \( \lambda \geq 0 \) has no solution.
Saddle point interpretation

- Max-min inequality for any $f$:
  \[ \sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z) \]

- Now, consider the optimal values of the primal and dual problems:

\[
p^* = \inf_{x} \sup_{\lambda \geq 0} L(x, \lambda) \geq \sup_{\lambda \geq 0} \inf_{x} L(x, \lambda)
\]
Other

- Definitions and examples
- Duality
- Classification of convex problems: LP, GP, SOCP, QCQP, etc.
- CVXPY & Python