Triangle meshes

Computer Graphics
CSE 167
Lecture 8
Examples

Spheres

Approximate sphere

Based on slides courtesy of Steve Marschner
Examples

Finite element analysis
Examples

Ottawa Convention Center
A small triangle mesh

12 triangles, 8 vertices
A large mesh

10 million triangles from a high-resolution 3D scan
About a trillion triangles from automatically processed satellite and aerial photography
Triangles

• Defined by three vertices
• Lies in the plane containing those vertices
• Normal of the plane is normal of the triangle
• Conventions (for this class; not everyone agrees):
  – Vertices are counter-clockwise as seen from the “outside” or “front”
  – Surface normal points towards the outside (“outward facing normals”)
Triangle meshes

• A bunch of triangles in 3D space that are connected together to form a surface

• Geometrically, a mesh is a piecewise planar surface
  – Almost everywhere, it is planar
  – Exceptions are at the edges where triangles join

• Often, it is a piecewise planar approximation of a smooth surface
  – In this case the creases between triangles are artifacts—we do not want to see them
Representation of triangle meshes

• Compactness
• Efficiency for rendering
  – Enumerate all triangles as triples of 3D points
• Efficiency of queries
  – All vertices of a triangle
  – All triangles around a vertex
  – Neighboring triangles of a triangle
  – Application dependent
    • Finding triangle strips
    • Computing subdivision surfaces
    • Mesh editing
Representations for triangle meshes

- Separate triangles
- Indexed triangle set
  - Shared vertices
- Triangle strips and triangle fans
  - Compression schemes for fast transmission
- Triangle-neighbor data structure
  - Supports adjacency queries
- Winged-edge data structure
  - Supports general polygon meshes
Separate triangles

<table>
<thead>
<tr>
<th>tris[0]</th>
<th>[0]</th>
<th>[1]</th>
<th>[2]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(x_0, y_0, z_0)</td>
<td>(x_2, y_2, z_2)</td>
<td>(x_1, y_1, z_1)</td>
</tr>
<tr>
<td>tris[1]</td>
<td>(x_0, y_0, z_0)</td>
<td>(x_3, y_3, z_3)</td>
<td>(x_2, y_2, z_2)</td>
</tr>
<tr>
<td></td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

Diagram of separate triangles with vertices \((x_0, y_0, z_0)\), \((x_1, y_1, z_1)\), \((x_2, y_2, z_2)\), and \((x_3, y_3, z_3)\).
Separate triangles

• Array of triples of points
  – float[\(n_T\)][3][3]: about 72 bytes per vertex
• 2 triangles per vertex (on average)
• 3 vertices per triangle
• 3 coordinates per vertex
• 4 bytes per coordinate (float)

• Various problems
  – Wastes space (each vertex stored 6 times)
  – Cracks due to roundoff
  – Difficult to find neighbors, if at all
Indexed triangle set

• Store each vertex once
• Each triangle points to its three vertices

Triangle {
    Vertex vertex[3];
}

Vertex {
    float position[3];  // or other data
}

// ... or ...

Mesh {
    float verts[nv][3];  // vertex positions (or other data)
    int tInd[nt][3];  // vertex indices
}
Indexed triangle set

<table>
<thead>
<tr>
<th>verts[0]</th>
<th>verts[1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0, y_0, z_0$</td>
<td>$x_1, y_1, z_1$</td>
</tr>
<tr>
<td>$x_2, y_2, z_2$</td>
<td>$x_3, y_3, z_3$</td>
</tr>
<tr>
<td>...</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>tInd[0]</th>
<th>tInd[1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 2, 1</td>
<td>0, 3, 2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Estimating storage space

- $n_T = \#\text{tris}; n_V = \#\text{verts}; n_E = \#\text{edges}$
- Euler: $n_V - n_E + n_T = 2$ for a simple closed surface
  - In general, sums to small integer
  - Argument for implication that $n_T:n_E:n_V$ is about 2:3:1
Indexed triangle set

• Array of vertex positions
  – float$[n_V][3]$: 12 bytes per vertex
    • (3 coordinates x 4 bytes) per vertex

• Array of triples of indices (per triangle)
  – int$[n_T][3]$: about 24 bytes per vertex
    • 2 triangles per vertex (on average)
    • (3 indices x 4 bytes) per triangle

• Total storage: 36 bytes per vertex (factor of 2 savings)

• Represents topology and geometry separately

• Finding neighbors is at least well defined
Data on meshes

• We often need to store additional information besides just the geometry
• Store additional data at faces, vertices, or edges
• Examples
  – Colors stored on faces, for faceted objects
  – Information about sharp creases stored at edges
  – Any quantity that varies continuously (without sudden changes, or discontinuities) gets stored at vertices
Key types of vertex data

• Surface normals
  – When a mesh is approximating a curved surface, store normals at vertices

• Texture coordinates
  – 2D coordinates that tell you how to paste images on the surface

• Positions
  – At some level this is just another piece of data
  – Position varies continuously between vertices
Differential geometry

• Tangent plane
  – At a point on a smooth surface in 3D there is a unique plane tangent to the surface called the tangent plane

• Normal vector
  – Vector perpendicular to a surface (that is, to the tangent plane)
  – Only unique for smooth surfaces (not at corners or edges)
Surface parameterization

• A surface in 3D is a two-dimensional entity
• Sometimes, we need 2D coordinates for points on the surface
• Defining these coordinates is parameterizing the surface
• Examples:
  – Cartesian coordinates on a rectangle (or other planar shape)
  – Cylindrical coordinates \((\theta, y)\) on a cylinder
  – Latitude and longitude on the Earth’s (ellipsoid) surface
  – Spherical coordinates \((\theta, \phi)\) on a sphere
Example: unit sphere

- Position:
  \[ x = \cos \theta \sin \phi \]
  \[ y = \sin \theta \]
  \[ z = \cos \theta \cos \phi \]

- Normal is also position
How to think about vertex normals

• Piecewise planar approximation converges pretty quickly to the smooth geometry as the number of triangles increases
  – For mathematicians: error is $O(h^2)$
• But the surface normals do not converge so well
  – Normal is constant over each triangle, with discontinuous jumps across edges
  – For mathematicians: error is only $O(h)$
• Better: store the “real” normal at each vertex, and interpolate to get normals that vary gradually across triangles
Interpolated normals—2D example

• Approximating circle with increasingly many segments
• Max error in position error drops by factor of 4 at each step
• Max error in normal only drops by factor of 2
Triangle strips

• Take advantage of the mesh property
  – Each triangle is usually adjacent to the previous
  – Let every vertex create a triangle by reusing the second and third vertices of the previous triangle
  – Every sequence of three vertices produces a triangle (but not in the same order)
  – e.g., 0, 1, 2, 3, 4, 5, 6, 7, ... leads to (0 1 2), (2 1 3), (2 3 4), (4 3 5), (4 5 6), (6 5 7), ...
  – For long strips, this requires about one index per triangle
Triangle strips

| verts[0] | \(x_0, y_0, z_0\) |
| verts[1] | \(x_1, y_1, z_1\) |
|          | \(x_2, y_2, z_2\) |
|          | \(x_3, y_3, z_3\) |
|          | \(:\) |

| tStrip[0] | 4, 0, 1, 2, 5, 8 |
| tStrip[1] | 6, 9, 0, 3, 2, 10, 7 |
|           | \(:\) |
Triangle strips

• Array of vertex positions
  – float[n_V][3]: 12 bytes per vertex
    • (3 coordinates x 4 bytes) per vertex

• Array of index lists
  – int[n_S][variable]: 2 + n indices per strip
  – On average, \((1 + \sum)\) indices per triangle (assuming long strips)
    • 2 triangles per vertex (on average)
    • About 4 bytes per triangle (on average)

• Total is 20 bytes per vertex (limiting best case)
  – Factor of 3.6 over separate triangles; 1.8 over indexed mesh
Triangle fans

• Same idea as triangle strips, but keep oldest rather than newest
  – Every sequence of three vertices produces a triangle
  – e.g., 0, 1, 2, 3, 4, 5, ... leads to (0 1 2), (0 2 3), (0 3 4), (0 4 5), ...
  – For long fans, this requires about one index per triangle

• Memory considerations exactly the same as triangle strip
Topology vs. geometry

• Two completely separate issues:
  – Mesh topology: how the triangles are connected (ignoring the positions entirely)
  – Geometry: where the triangles are in 3D space
Topology/geometry examples

• Same geometry, different mesh topology

• Same mesh topology, different geometry
Validity of triangle meshes

• In many cases we care about the mesh being able to bound a region of space nicely
• In other cases we want triangle meshes to fulfill assumptions of algorithms that will operate on them (and may fail on malformed input)
• Again, two completely separate issues
  – Topology: how the triangles are connected (ignoring the positions entirely)
  – Geometry: where the triangles are in 3D space
Topological validity

• Strongest property: be a manifold
  – This means that no points should be "special"
  – Interior points are fine
  – Edge points: each edge must have exactly 2 triangles
  – Vertex points: each vertex must have one loop of triangles

• Slightly looser: manifold with boundary
  – Weaken rules to allow boundaries
Topological validity

• Consistent orientation
  – Which side is the “front” or “outside” of the surface and which is the “back” or “inside”?
  – Rule: you are on the outside when you see the vertices in counter-clockwise order
  – In mesh, neighboring triangles should agree about which side is the front!
  – Warning: not always possible

- non-orientable
Geometric validity

• Generally, want non-self-intersecting surface
• In general, this is hard to guarantee
  – Far-apart parts of mesh might intersect
Triangle neighbor structure

• Extension to indexed triangle set
• Triangle points to its three neighboring triangles
• Vertex points to a single neighboring triangle
• Can now enumerate triangles around a vertex
Triangle neighbor structure

Triangle {
    Triangle nbr[3];
    Vertex vertex[3];
}

// t.neighbor[i] is adjacent
// across the edge from i to i+1

Vertex {
    // ... per-vertex data ...
    Triangle t;  // any adjacent tri
}

// ... or ...

Mesh {
    // ... per-vertex data ...
    int tInd[nt][3];  // vertex indices
    int tNbr[nt][3];  // indices of neighbor triangles
    int vTri[nv];  // index of any adjacent triangle
}
Triangle neighbor structure

<table>
<thead>
<tr>
<th>vTri[0]</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>vTri[1]</td>
<td>6</td>
</tr>
<tr>
<td>vTri[2]</td>
<td>1</td>
</tr>
<tr>
<td>vTri[3]</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>tNbr[0]</th>
<th>1, 6, 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>tNbr[1]</td>
<td>10, 2, 0</td>
</tr>
<tr>
<td>tNbr[2]</td>
<td>3, 1, 12</td>
</tr>
<tr>
<td>tNbr[3]</td>
<td>2, 13, 4</td>
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<td>...</td>
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CSE 167, Winter 2020
Triangle neighbor structure

TrianglesOfVertex(v) {
    t = v.t;
    do {
        find t.vertex[i] == v;
        t = t.nbr[pred(i)];
    } while (t != v.t);
}

pred(i) = (i+2) % 3;
succ(i) = (i+1) % 3;
Triangle neighbor structure

• Indexed mesh was 36 bytes per vertex
• Add an array of triples of indices (per triangle)
  – int[n_T][3]: about 24 bytes per vertex
    • 2 triangles per vertex (on average)
    • (3 indices x 4 bytes) per triangle
• Add an array of representative triangle per vertex
  – int[n_V]: 4 bytes per vertex
• Total storage: 64 bytes per vertex
  – Still not as much as separate triangles
Triangle neighbor structure—refined

Triangle {
    Edge nbr[3];
    Vertex vertex[3];
}

// if t.nbr[i].i == j
// then t.nbr[i].t.nbr[j] == t

Edge {
    // the i-th edge of triangle t
    Triangle t;
    int i; // in {0,1,2}
    // in practice t and i share 32 bits
}

Vertex {
    // ... per-vertex data ...
    Edge e; // any edge leaving vertex
}

T0.nbr[0] = { T1, 2 }
T1.nbr[2] = { T0, 0 }
V0.e = { T1, 0 }
Triangle neighbor structure

TrianglesOfVertex(v) {
    {t, i} = v.e;
    do {
        {t, i} = t.nbr[pred(i)];
    } while (t != v.t);
}

pred(i) = (i+2) % 3;
succ(i) = (i+1) % 3;

T0.nbr[0] = { T1, 2 }
T1.nbr[2] = { T0, 0 }
V0.e = { T1, 0 }
Winged-edge mesh

• Edge-centric rather than face-centric
  – Therefore also works for polygon meshes

• Each (oriented) edge points to:
  – Left and right forward edges
  – Left and right backward edges
  – Front and back vertices
  – Left and right faces

• Each face or vertex points to
  one edge
Winged-edge mesh

Edge {
    Edge hl, hr, tl, tr;
    Vertex h, t;
    Face l, r;
}

Face {
    // per-face data
    Edge e; // any adjacent edge
}

Vertex {
    // per-vertex data
    Edge e; // any incident edge
}
Winged-edge structure

\[
\text{EdgesOfFace}(f) \{ \\
\quad e = f.e; \\
\quad \text{do} \{ \\
\qquad \text{if (e.t == } f) \\
\qquad \quad e = e.hl; \\
\qquad \text{else} \\
\qquad \quad e = e.hr; \\
\qquad \}\text{ while (e != } f.e); \\
\}
\]

\[
\begin{array}{c|cccc}
\text{edge[0]} & \text{hl} & \text{hr} & \text{tl} & \text{tr} \\
1 & 4 & 2 & 3 & \\
\text{edge[1]} & 18 & 0 & 16 & 2 \\
\text{edge[2]} & 12 & 1 & 3 & 0 \\
& \vdots & & & \\
\end{array}
\]
Winged-edge structure

• Array of vertex positions: 12 bytes/vert
• Array of 8-tuples of indices (per edge)
  – Head/tail left/right edges + head/tail verts + left/right tris
  – int$[n_E][8]$: about 96 bytes per vertex
    • 3 edges per vertex (on average)
    • (8 indices x 4 bytes) per edge
• Add a representative edge per vertex
  – int$[n_V]$: 4 bytes per vertex
• Total storage: 112 bytes per vertex
  – But it is cleaner and generalizes to polygon meshes
Winged-edge optimizations

- Omit faces if not needed
- Omit one edge pointer on each side
  - Results in one-way traversal
Half-edge structure

• Simplifies, cleans up winged edge
  – Still works for polygon meshes
• Each half-edge points to:
  – Next edge (left forward)
  – Next vertex (front)
  – The face (left)
  – The opposite half-edge
• Each face or vertex points to one half-edge
Half-edge structure

HEdge {
    HEdge pair, next;
    Vertex v;
    Face f;
}

Face {
    // per-face data
    HEdge h;  // any adjacent h-edge
}

Vertex {
    // per-vertex data
    HEdge h;  // any incident h-edge
}
Half-edge structure

EdgesOfFace\(v\) { 
    h = \! v.h;
    do {
        h = h.next.pair;
    } while (h != \! v.h);
}

<table>
<thead>
<tr>
<th>pair</th>
<th>next</th>
</tr>
</thead>
<tbody>
<tr>
<td>hedge[0]</td>
<td>1</td>
</tr>
<tr>
<td>hedge[1]</td>
<td>0</td>
</tr>
<tr>
<td>hedge[2]</td>
<td>3</td>
</tr>
<tr>
<td>hedge[3]</td>
<td>2</td>
</tr>
<tr>
<td>hedge[4]</td>
<td>5</td>
</tr>
<tr>
<td>hedge[5]</td>
<td>4</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
Half-edge structure

• Array of vertex positions: 12 bytes/vert
• Array of 4-tuples of indices (per h-edge)
  – Next, pair h-edges + head vert + left tri
  – int[2n_E][4]: about 96 bytes per vertex
    • 6 h-edges per vertex (on average)
    • (4 indices x 4 bytes) per h-edge
• Add a representative h-edge per vertex
  – int[n_V]: 4 bytes per vertex
• Total storage: 112 bytes per vertex
Half-edge optimizations

- Omit faces if not needed
- Use implicit pair pointers
  - They are allocated in pairs
  - They are even and odd in an array