1. It is better to be player B. Here is the strategy for player B that guarantees that his probability of winning will always be greater than that of player A.

- If player A chooses spinner ⃝a, then player B chooses spinner ⃝c.
- If player A chooses spinner ⃝b, then player B chooses spinner ⃝a.
- If player A chooses spinner ⃝c, then player B chooses spinner ⃝b.

Let us show that the probability of winning for player A is less than 1/2 in all the three cases. The probability that spinner ⃝a defeats spinner ⃝c is

\[ P(a = 9) + P(a = 5, c = 2) = \frac{1}{3} + \left(\frac{1}{3}\right)^2 = \frac{4}{9} < \frac{1}{2} \]

The probability that spinner ⃝b defeats spinner ⃝a is

\[ P(a = 1) + P(a = 5, b = 8) = \frac{1}{3} + \left(\frac{1}{3}\right)^2 = \frac{4}{9} < \frac{1}{2} \]

The probability that spinner ⃝c defeats spinner ⃝b is

\[ P(b = 3, c = 7) + P(b = 4, c = 7) + P(b = 3, c = 6) + P(b = 4, c = 6) = 4 \cdot \left(\frac{1}{3}\right)^2 = \frac{4}{9} < \frac{1}{2} \]

Thus player B has probability 5/9 of winning in all three cases. What makes this possible is that the situation is non-transitive: with probability 5/9, ⃝a beats ⃝b, ⃝b beats ⃝c, and ⃝c beats ⃝a. Thus given any spinner, there is always one spinner that is worse and another that is better.

2. (a) If A is an event independent of itself, then \( P(A \cap A) = P(A)P(A) \) by the definition of stochastic independence. But clearly \( A \cap A = A \), so \( P(A \cap A) = P(A) \). Now let us denote \( P(A) \) by \( x \). From the two observations above, we find that \( x = x^2 \), which is equivalent to \( x^2 - x = x(x - 1) = 0 \). The only roots of this equation are \( x = 0 \) and \( x = 1 \).
(b) First, write \( P(A \cup B) = P(A) + P(B) - P(AB) \). If \( A \) and \( B \) are independent events, then \( P(AB) = P(A)P(B) \) and so

\[
P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.3 + 0.4 - (0.3)(0.4) = 0.58
\]

If \( A \) and \( B \) are disjoint events, then \( P(AB) = 0 \). Hence \( P(A \cup B) = P(A) + P(B) = 0.7 \). If \( P(A) \) were 0.6 and \( P(B) \) were 0.8 then the events could be independent, but they could not be disjoint. Assuming the events \( A \) and \( B \) are disjoint, we conclude that \( P(AB) = 0 \) and therefore \( P(A \cup B) = P(A) + P(B) = 0.6 + 0.8 = 1.4 \). But this is a contradiction, since the probability of an event cannot be greater than 1.

3. Let \( W \) and \( F \) be the events that component 1 works and that the system functions, respectively. Then, using the Bayes inversion rule, we find that

\[
P(W|F) = \frac{P(F|W)P(W)}{P(F)} = \frac{1 \cdot \frac{1}{2}}{1 - P(F^c)} = \frac{\frac{1}{2}}{1 - (\frac{1}{2})^n} = \frac{2^{n-1}}{2^n - 1}
\]

4. The statement is false, and here is a counter-example. Consider the sample space consisting of two independent tosses of a fair coin, namely \( \Omega = \{HH, TT, HT, TH\} \). Define the events

\[
A = \{HH, HT\}, \quad B = \{TT, HT\}, \quad \text{and} \quad E = \{HH, TT, HT\}
\]

Then

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2} = P(A)
\]

so events \( A \) and \( B \) are independent. Since \( E = A \cup B \), we have \( A \cap E = A \) and \( B \cap E = B \). Hence \( P(A|E) = P(A)/P(E) = 2/3 \) and \( P(B|E) = P(B)/P(E) = 2/3 \). Yet \( A \cap B = \{HT\} \) and \( AB \cap E = \{HT\} \). Therefore

\[
P(AB|E) = \frac{P(\{HT\})}{P(E)} = \frac{1}{3} \neq \frac{4}{9} = P(A|E) \cdot P(B|E)
\]

Thus events \( A \), \( B \) are independent, but they are not conditionally independent given the event \( E \).

5. It can be easily seen by symmetry that \( P(A) = P(B) = P(C) = 1/2 \). Moreover, observe that \( A \cap B = A \cap C = B \cap C = \{HH\} \). Therefore \( P(AB) = P(AC) = P(BC) = 1/4 \), and the three events are pairwise independent. However

\[
P(ABC) = P(\{HH\}) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}
\]
6. (a) Let $A$ denote the event that a single roll of dice results in a win (for either player). Thus $P(A) = p$ and $P(A^c) = 1 - p = q$. Then Alice wins in the following sequences of outcomes: $A \cdot \cdot \cdot , A^c A^c A \cdot \cdot \cdot , A^c A^c A^c A A \cdot \cdot \cdot$, and so on. It follows that the probability that Alice wins is given by

$$P(Alice) = p + q^2p + q^4p + \cdots = p \sum_{i=0}^{\infty} (q^2)^i = \frac{p}{1 - q^2} = \frac{1}{2 - p}$$

Now $P(Bob) = 1 - P(Alice) = (1 - p)/(2 - p)$. Another way to solve this problem is to observe that Bob wins in the following sequences of outcomes: $A^c A \cdot \cdot \cdot , A^c A^c A^c A^c A \cdot \cdot \cdot$, and so on. Thus $P(Bob) = qP(Alice) = 1 - P(Alice)$. Solving this for $P(Alice)$ immediately produces $P(Alice) = 1/(1 + q) = 1/(2 - p)$.

(b) The second method of solution above is much more convenient in this case. Let $E_i$ be the event that player $P_i$ wins the game. Observe that $P(E_i) = q^{i-1}P(E_1)$ for all $i = 1, 2, \ldots k$. Since some player must win, we have

$$1 = \sum_{i=1}^{k} P(E_i) = \left(1 + q + q^2 + \cdots + q^{k-1}\right) P(E_1) = \frac{1 - q^k}{1 - q} P(E_1)$$

This immediately produces $P(E_1) = (1 - q)/(1 - q^k)$ and $P(E_i) = q^{i-1}(1 - q)/(1 - q^k)$ for all $i = 1, 2, \ldots, k$. In terms of $p$, this is $P(E_i) = p/(1 - p)^{i-1}/(1 - (1-p)^k)$.

(c) Here, let $A$ denote the event that Alice wins on her roll and let $B$ denote the event that Bob wins on his roll. Thus $P(A) = p_1$ and $P(B) = p_2$. Further, let $q_1 = 1 - p_1 = P(A^c)$ and $q_2 = 1 - p_2 = P(B^c)$. Then Alice wins in the following sequences of outcomes: $A \cdot \cdot \cdot , A^c B^c A \cdot \cdot \cdot , A^c B^c A^c A^c A \cdot \cdot \cdot$, and so on. Hence the probability that Alice wins is given by

$$P(Alice) = p_1 + q_1q_2p_1 + q_1q_2q_1q_2p + \cdots = p_1 \sum_{i=0}^{\infty} (q_1q_2)^i = \frac{p_1}{1 - q_1q_2}$$

The probability that Bob wins is given by $P(Bob) = 1 - P(Alice)$. Altogether, we find that $P(Alice) = p_1/(p_1 + p_2 - p_1p_2)$ and $P(Bob) = p_2(1 - p_1)/(p_1 + p_2 - p_1p_2)$.

7. For $i = 1, 2$, let $A_i$ denote the event that none of the $N$ trials result in the outcome $i$. The event we are after is that outcome 1 occurs at least once and outcome 2 occurs at least once, which can be expressed as $A_1^c \cap A_2^c = (A_1 \cup A_2)^c$. Now

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = (1 - p_1)^N + (1 - p_2)^N - p_0^N$$

where we have used the fact that the event $A_1 \cap A_2$ corresponds to the single sequence of outcomes $0 \cdots 0$. It follows that the desired probability is

$$P(A_1^c \cap A_2^c) = 1 - P(A_1 \cup A_2) = 1 - (1 - p_1)^N - (1 - p_2)^N + p_0^N$$