1. A number is chosen uniformly at random in the interval $[0, 1]$. Find the probability density function of the negative of its natural logarithm. That is, if $X \sim U(0, 1)$, what is the p.d.f. of $-\ln X$?

2. Suppose that $X$ is a continuous RV with CDF $F_X(u)$ and probability density function $f_X(u)$. Let

$$Y = \alpha |X| + \beta \quad \text{and} \quad Z = \text{sign}(X)$$

where $\alpha, \beta$ are arbitrary real numbers with $\alpha \neq 0$, while $\text{sign}(\cdot)$ is the sign function defined by $\text{sign}(u) = +1$ if $u \geq 0$ and $\text{sign}(u) = -1$ if $u < 0$.

(a) Determine the CDF and the p.d.f. or p.m.f. of $Y$ and $Z$ in terms of the CDF and p.d.f. of $X$.

(b) Now assume that $X \sim N(0, 9)$. Find the CDF and the p.d.f./p.m.f. of $Y$ and $Z$ in this case?

3. Let $X$ be an arbitrary continuous random variable with CDF $F_X(u)$.

(a) Define another random variable $Y$ by $Y = F_X(X)$. Show that, regardless of what $F_X(u)$ is, $Y$ is uniformly distributed over the interval $[0, 1]$.

(b) Now let’s do the reverse. Suppose we wish to generate a continuous random variable $X$ with a specified distribution. Namely, we want the CDF of $X$ to be some specified function $F(u)$. However, all we are given is the random variable $Y$ that is uniform on $[0, 1]$. Define $X$ by $X = F^{-1}(Y)$, then show that $X$ has the desired CDF $F(u)$. (This is a widely used method to generate random variables of desired distributions, for example in computer simulations.)

4. Suppose that two cards are drawn at random from a deck of 52 cards. Let $X$ be the number of aces obtained and let $Y$ be the number of queens obtained.

(a) Find the joint probability mass function of $X$ and $Y$.

(b) Find the marginal probability mass function of $X$ and that of $Y$.

For those who never play card games: there are 4 aces and 4 queens in a deck of 52 cards.
5. The discrete random variables $X$ and $Y$ have joint p.m.f. $p_{X,Y}(u,v)$ given by

<table>
<thead>
<tr>
<th>$v \backslash u$</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{12}$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{12}$</td>
<td>0</td>
<td>$\frac{1}{12}$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>$\frac{1}{12}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

(a) Find the joint CDF of $X$ and $Y$. Specify the value of $F_{X,Y}(u,v)$ for all $u$ and $v$.
(b) Find the marginal probability mass functions $p_X(u)$ and $p_Y(v)$ of $X$ and $Y$.
(c) Find $P\{X \leq Y\}$ and $P\{X + Y \leq 8\}$

6. Is the following function

$$F(u,v) = \begin{cases} 
0 & \text{if } u + v < 1 \\
1 & \text{if } u + v \geq 1 
\end{cases}$$

a valid joint CDF. Why or why not? Prove your answer and show your work.