Problem 1 (26 points)

a. The following are statements about events $A$, $B$, $C$ with probabilities $P(A)$, $P(B)$, $P(C)$ that are all nonzero and strictly less than 1. The events are arbitrary, unless otherwise specified.

True False

✓    □  $P(A \oplus B) = P(A|B^c)P(B^c) + P(A^c|B)P(B)$
✓    □  if $P(A) = P(B)$, then $P(A|B) = P(B|A)$
✓    □  if $P(A) + P(B) - P(A \cup B) = P(A)P(B)$, then $A$ and $B$ are independent
□    ✓  if $P(ABC) = P(A)P(B)P(C)$, then $A$, $B$, $C$ are independent events

b. The three events $A$, $B$, $C$ are independent. It is known that $P(A) = 0.5$, $P(B) = 0.4$, and $P(ABC) = 0.1$. Determine the probabilities of the eight events $ABC$, $ABC^c$, $AB^cC$, $AB^cC^c$, $A^cBC$, $A^cB^cC$, $A^cB^cC$, $A^cB^cC$ in the Karnaugh map below.

Answer: Since the three events are independent, we have $0.1 = P(ABC) = P(A)P(B)P(C)$. From this, we find that $P(C) = 0.1/(0.5 \cdot 0.4) = 0.5$. We can now use the independence of the three events again to compute $P(A^*B^*C^*)$ as $P(A^*)P(B^*)P(C^*)$ where, for an event $E$, the notation $E^*$ is used to denote either $E$ or $E^c$. This produces the following probabilities:
c. In the following statements, \( X \) is a generic continuous random variable. The functions \( F_X(u) \) and \( F_Y(u) \) are the cumulative distribution functions (CDFs) of \( X \) and \( Y \), respectively.

**True**  **False**
- \( P\left(X^2 - 2X + 1 < 0\right) = 0 \)
- \( \square \) if \( Y = |X| \), then \( F_Y(u) = F_X(u) + F_X(-u) \) for all real \( u > 0 \)

**Problem 2** (20 points)

A box contains 2 black balls and 6 white balls. First, a ball is drawn at random from the box, its color is observed, and the ball is returned to the box along with 4 additional balls of the other color (if the ball drawn was black, then 4 white balls are added to the box; if the ball was white, then 4 black balls are added to the box). After this is done, a second ball is drawn at random from the box.

a. What is the probability that the second ball drawn is black?

**Answer:** Let \( A \) denote the event that the *first* ball drawn was black. Since there are 2 black balls and 6 white balls in the box, it is clear that \( P(A) = 2/(2+6) = 1/4 \) and \( P(A^c) = 3/4 \). Let \( B \) be the event that the *second* ball drawn is black. By the theorem of total probability, we have:

\[
P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)
\]

If \( A \) occurred then, when the second ball is drawn, there are 2 black balls and \( 6 + 4 = 10 \) white balls in the box. Hence \( P(B|A) = 2/(2+10) = 1/6 \). If \( A \) did not occur then, when the second ball is drawn, there are \( 2 + 4 = 6 \) black balls and 6 white balls. Hence \( P(B|A^c) = 1/2 \). Putting all of this together, we compute

\[
P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = \frac{1}{6} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{5}{12}
\]

**Probability that the second ball drawn is black = \( \frac{5}{12} \)**

b. What is the probability that the first ball was black, given that the second ball drawn is black?

**Answer:** By the Bayes rule, we have

\[
P(A|B) = \frac{P(B|A)P(A)}{P(B)}
\]

In part (a), we have computed \( P(B|A) = 1/6 \), \( P(A) = 1/4 \), and \( P(B) = 5/12 \). Hence

\[
P(A|B) = \frac{(1/6) \cdot (1/4)}{5/12} = \frac{1}{10}
\]

**Probability that the first ball was black, given that the second ball is black = \( \frac{1}{10} \)**
Problem 3 (30 points)

An experiment consists of tossing a biased coin three times. The probability that the coin turns heads is \( p \) and the probability that it turns tails is \( q = 1 - p \). Express your answers in terms of \( p \) and \( q \).

a. Let \( X \) be the number of heads observed on the three tosses. Compute the probability mass function of \( X \) and the cumulative distribution function (CDF) of \( X \).

**Answer:** Using the theory of independent trials, we find that

\[
P(X = k) = \binom{3}{k} p^k q^{3-k} \quad \text{for } k = 0, 1, 2, 3
\]

and \( P(X = k) = 0 \) otherwise. This gives the following probability mass function:

\[
p_X(u) = \begin{cases} 
q^3 & u = 0 \\
3pq^2 & u = 1 \\
3p^2q & u = 2 \\
p^3 & u = 3 \\
0 & \text{otherwise}
\end{cases}
\]

The CDF of \( X \) can be now computed as follows: \( F_X(u) = \sum_{k=-\infty}^{[u]} p_X(k) \). Hence

\[
F_X(u) = \begin{cases} 
0 & u < 0 \\
q^3 & 0 \leq u < 1 \\
q^3 + 3pq^2 & 1 \leq u < 2 \\
1 - p^3 & 2 \leq u < 3 \\
1 & u \geq 3
\end{cases}
\]

b. Let \( Y \) be the number of consecutive heads observed on the three tosses. For example, if the outcomes are \( HHT \) or \( THH \) then \( Y = 1 \), while if the outcomes are \( HHT \) or \( THH \) then \( Y = 2 \). Compute the probability mass function of \( Y \).

**Answer:** Let us consider the eight possible outcomes of the experiment, their probability, and the corresponding values of \( X \) and \( Y \). The information can be organized in tabular form:

\[
\begin{align*}
HHH: & \quad p^3 \quad X = 3, Y = 3 \\
HHT: & \quad p^2q \quad X = 2, Y = 2 \\
HTH: & \quad p^2q \quad X = 2, Y = 1 \\
HTT: & \quad pq^2 \quad X = 2, Y = 1 \\
THH: & \quad p^2q \quad X = 2, Y = 2 \\
THT: & \quad pq^2 \quad X = 2, Y = 1 \\
TTH: & \quad pq^2 \quad X = 1, Y = 1 \\
TTT: & \quad q^3 \quad X = 0, Y = 0
\end{align*}
\]

From this, we immediately obtain the probability mass function of \( Y \), as follows:

\[
p_Y(u) = \begin{cases} 
q^3 & u = 0 \\
3pq^2 + p^2q & u = 1 \\
2p^2q & u = 2 \\
p^3 & u = 3 \\
0 & \text{otherwise}
\end{cases}
\]
c. Now suppose that the experiment is repeated until the condition $X > Y$ is observed. What is the probability that it takes exactly $k$ repetitions of the experiment to observe this condition for the first time?

**Answer:** It can be seen from (1) that there is only one outcome for which $X > Y$, namely $HTH$. The probability of this outcome is $p^2q$. In general, the probability that a basic event $A$ occurs for the first time on the $k$-th trial is given by

$$(1 - P(A))^{k-1}P(A)$$

as was shown in class. In this case, the event of interest is that the outcome $HTH$ occurs, which happens with probability $P(A) = p^2q$. Thus the answer is $p^2q(1 - p^2q)^{k-1}$.

- **Probability that the experiment will be repeated exactly $k$ times**

$$p^2q(1 - p^2q)^{k-1}$$

**Problem 4 (24 points)**

The daily demand for gas (counted in thousands of gallons) at a gas station is a continuous random variable $X$, whose probability density function is given by

$$f_X(u) = \begin{cases} \alpha(1-u)^4 & \text{if } 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Determine the value of $\alpha$.

**Answer:** To compute $\alpha$, we use one of the fundamental properties of probability density functions, namely

$$1 = \int_{-\infty}^{\infty} f_X(u) \, du = \int_{0}^{1} \alpha(1-u)^4 \, du = \alpha \left. \frac{1}{5}(1-u)^5 \right|_{0}^{1} = \frac{\alpha}{5}$$

From this, it follows that $\alpha = 5$.

- **$\alpha = 5$**

b. The station manager would like to install a central tank that will hold all of the station’s gas, and will be filled-up every morning. Determine the minimum required capacity $c$ of the tank (in thousands of gallons) such that the probability that the station runs out of gas by the end of the day — namely $P(X > c)$ — is at most $10^{-5}$.

**Answer:** We need to find the minimum value of $c$ which satisfies $P(X > c) \leq 10^{-5}$. First, let us compute $P(X > c)$ by integrating the probability density function, as follows:

$$P(X > c) = \int_{c}^{\infty} f_X(u) \, du = 5 \int_{c}^{1} (1-u)^4 \, du = 5 \left( \frac{1}{5}(1-u)^5 \right|_{c}^{1} \right) = (1-c)^5$$

Now

$$(1-c)^5 \leq 10^{-5} \iff 1-c \leq 10^{-1} \iff c \geq 0.9$$

Thus the minimum value of $c$ which satisfies $P(X > c) \leq 10^{-5}$ is $c = 0.9$.

- **$c = 0.9$**