INSTRUCTIONS

The exam consists of 4 problems worth a total of 100 points. Write your answers in the spaces provided. Show all your work, except in the true/false problems. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem.

Good luck!
Problem 1 (26 points)

a. The following are statements about events $A$, $B$, $C$ with probabilities $P(A)$, $P(B)$, and $P(C)$ that are all nonzero and strictly less than 1. The events are arbitrary, unless otherwise specified. Answer True if the statement is true for all such events $A$, $B$, $C$. Otherwise, answer False.

True False

☐ ☐ $P(A \oplus B) = P(A|B^c)P(B^c) + P(A^c|B)P(B)$

☐ ☐ if $P(A) = P(B)$, then $P(A|B) = P(B|A)$

☐ ☐ if $P(A) + P(B) - P(A \cup B) = P(A)P(B)$, then $A$ and $B$ are independent

☐ ☐ if $P(ABC) = P(A)P(B)P(C)$, then $A$, $B$, $C$ are independent events

Check the appropriate box for each of the four statements above. No justification is required. However, so as not to give points for random guessing, you will receive +3 points for correct answer, 0 points for no answer, and −3 points for wrong answer.
b. The three events $A$, $B$, $C$ are independent. It is known that $P(A) = 0.5$, $P(B) = 0.4$, and $P(ABC) = 0.1$. Determine the probabilities of the eight events $ABC$, $ABC^c$, $AB^cC$, $AB^cC^c$, $A^cBC$, $A^cBC^c$, $A^cB^cC$, $A^cB^cC^c$ in the Karnaugh map below.

Write your answers directly in the Karnaugh map above. No justification is required. Grading: 1 point for each correct answer, 0 points for a wrong answer or no answer.
c. In the following statements, $X$ is a generic continuous random variable. The functions $F_X(u)$ and $F_Y(u)$ are the cumulative distribution functions (CDFs) of $X$ and $Y$, respectively.

True False

☐ ☐ $P\left(X^2 - 2X + 1 < 0\right) = 0$

☐ ☐ If $Y = |X|$, then $F_Y(u) = F_X(u) + F_X(-u)$ for all real $u > 0$

Check the appropriate box for each of the two statements above. **No justification is required.** However, so as not to give points for random guessing, you will receive +3 points for correct answer, 0 points for no answer, and −3 points for wrong answer.
Problem 2 (20 points)

A box contains 2 black balls and 6 white balls. First, a ball is drawn at random from the box, its color is observed, and the ball is returned to the box along with 4 additional balls of the other color (if the ball drawn was black, then 4 white balls are added to the box; if the ball was white, then 4 black balls are added to the box). After this is done, a second ball is drawn at random from the box.

a. What is the probability that the second ball drawn is black?

\[
\text{Probability that the second ball drawn is black} = \quad \text{[Expression]} \quad \text{[Expression]} \quad \text{[Expression]}
\]
b. What is the probability that the first ball was black, given that the second ball drawn is black?
Problem 3 (30 points)

An experiment consists of tossing a biased coin three times. The probability that the coin turns heads is \( p \) and the probability that it turns tails is \( q = 1 - p \). Express your answers in terms of \( p \) and \( q \).

a. Let \( X \) be the number of heads observed on the three tosses. Compute the probability mass function of \( X \) and the cumulative distribution function (CDF) of \( X \).

\[
\begin{align*}
p_X(u) &= \begin{cases} 
\text{ } & 
\text{ } \\
\end{cases} \\
F_X(u) &= \begin{cases} 
\text{ } & 
\text{ } 
\end{cases}
\end{align*}
\]
b. Let $Y$ be the number of **consecutive** heads observed on the three tosses. For example, if the outcomes are $HTT$ or $HTH$ then $Y = 1$, while if the outcomes are $HHT$ or $THH$ then $Y = 2$. Compute the probability mass function of $Y$. 

\[
p_Y(u) = \begin{cases} 
\frac{1}{8} & \text{if } u = 0 \\
\frac{3}{8} & \text{if } u = 1 \\
\frac{3}{8} & \text{if } u = 2 \\
\frac{1}{8} & \text{if } u = 3
\end{cases}
\]
c. Now suppose that the experiment is repeated until the condition $X > Y$ is observed. What is the probability that it takes exactly $k$ repetitions of the experiment to observe this condition for the first time?

Probability that the experiment will be repeated exactly $k$ times =
Problem 4 (24 points)

The daily demand for gas (counted in thousands of gallons) at a gas station is a continuous random variable $X$, whose probability density function is given by

$$f_X(u) = \begin{cases} 
\alpha(1 - u)^4 & \text{if } 0 \leq u \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

a. Determine the value of $\alpha$.

\[ \alpha = \]  

b. The station manager would like to install a central tank that will hold all of the station’s gas, and will be filled-up every morning. Determine the minimum required capacity $c$ of the tank (in thousands of gallons) such that the probability that the station runs out of gas by the end of the day — namely $P(X > c)$ — is at most $10^{-5}$.

\[ c = \]