INSTRUCTIONS

The exam consists of 7 problems worth a total of 325 points. Of these, 25 points in Problem 3 are **extra-credit points**. That is, a score of only 300 points is needed to receive the full 50% credit given to the final exam. Scores in excess of 300 points will compensate for your homework grades.

Write your answers in the spaces provided. Show *all* of your work, except in the first two problems where no justification is required. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem.

**Good luck!**
Problem 1 (96 points)

Check the appropriate box for each of the statements below. No justification is required. Grading is as follows: 6 points for correct answer, 3 points for no answer, and 0 points for wrong answer.

a. The following are statements about an arbitrary continuous random variable $X$ with finite mean and variance. Answer True only if the statement is true for all such random variables.

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
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| ☐    | ☐     | Let $Y = X^2$. If $E[Y] = 0$, then $\text{Var}(X) = 0$
| ☐    | ☐     | Let $Y = X - E[X]$. Then $\text{Var}(Y) = \text{Var}(X)$
| ☐    | ☐     | Let $Y = e^X$. Then $F_Y(v) = F_X(\ln v)$ for all $v \in \mathbb{R}$
| ☐    | ☐     | Let $Y = \ln X$. Then $f_Y(v) = v f_X(e^v)$ for all $v \in \mathbb{R}$

b. The following are statements about jointly continuous random variables $X, Y$ with joint probability density function $f_{X,Y}(u, v)$ given by

$$f_{X,Y}(u, v) = \begin{cases} 
  e^{-(u+v)} & u \geq 0 \text{ and } v \geq 0 \\
  0 & \text{otherwise}
\end{cases}$$

<table>
<thead>
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<th>True</th>
<th>False</th>
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</table>
| ☐    | ☐     | The random variables $X$ and $Y$ are independent
| ☐    | ☐     | $P\{X \leq Y\} = 0.5$
| ☐    | ☐     | $E[Y^2] = \int_0^\infty \int_0^\infty v^2 e^{-(u+v)} \, du \, dv$
| ☐    | ☐     | $X$ is an exponential random variable with parameter $\lambda = 1$. 

c. The following are statements about arbitrary jointly continuous random variables $X, Y$, with joint probability density function $f_{X,Y}(u,v)$, that have finite mean and variance. Answer True only if the statement is true for all such random variables.

True  False  
☐  ☐  $P\{ Y \leq X \leq 1-Y \} = \int_{-\infty}^{1/2} \int_{v}^{1-v} f_{X,Y}(u,v) \, du \, dv$

☐  ☐  Let $Z = X + Y$. Then $f_Z(w) = f_X(u) \ast f_Y(v)$ for all $w$, where $\ast$ denotes the convolution operation

☐  ☐  If $X$ and $Y$ are independent, then 
$$P\{XY > 0\} = P\{X > 0\} \, P\{Y > 0\} + P\{X < 0\} \, P\{Y < 0\}$$

☐  ☐  If $X, Y$ are uncorrelated, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

d. The following are statements about arbitrary jointly continuous random variables $X, Y$ with finite mean and variance. Answer True only if the statement is true for all such random variables.

True  False  
☐  ☐  If $\text{Cov}(X, Y) = 0$, then $X$ and $Y$ are independent

☐  ☐  If $\text{Cov}(X, Y) = 3$, then $\text{Cov}(2X - 5, 4Y + 2) = 24$

☐  ☐  If $X$ and $Y$ are independent, then $\text{Cov}(X + Y, X) = \text{Var}(X)$

☐  ☐  If $X$ and $Y$ are uncorrelated, then $\text{Cov}(X - Y, X) = \text{Var}(X)$
Problem 2 (40 points)

The following are multiple choice questions: check a single box for each. **No justification is required.** Grading: 8 points for correct answer, 2 points for no answer, and 0 points for wrong answer.

1. Alice, Bob, Carol, and David put notes with their names in a box. Then they draw notes from the box at random, without replacement: first Alice, then Bob, then Carol, then David. Let \( X \) be the number of people that drew the note with their own name on it. The expected value of \( X \) is
   
   □ 1
   □ 2
   □ 3
   □ none of the above

2. You are offered to play the following game. Two fair dice are rolled. If the two dice show the same number, you win $6. Otherwise, you loose $1. You can repeat the game as many times as you want.
   
   □ On average, playing this game will be profitable for you.
   □ On average, playing this game will be unprofitable for you.
   □ On average, you will neither win nor loose by playing this game.
   □ The outcome of the game is random, and nothing can be concluded on average.

3. Let \( X \) be a Poisson random variable with parameter \( \lambda \). Then the expected value of the random variable \( Y = X^2 + X + 1 \) is given by
   
   □ \( \lambda^2 + \lambda + 1 \)
   □ \( (\lambda + 1)^2 \)
   □ \( 2\lambda + 1 \)
   □ none of the above
4. Let $X$ and $Y$ be independent Gaussian random variables, with means $\mu_X$ and $\mu_Y$ and variances $\sigma_X^2$ and $\sigma_Y^2$, respectively. Then $Z = X + Y$ is a Gaussian random variable, whose mean and variance are given by

- $\mu_X \mu_Y, \sigma_X^2 \sigma_Y^2$
- $\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2$
- $\max\{\mu_X, \mu_Y\}, \max\{\sigma_X^2, \sigma_Y^2\}$
- $\mu_X + \mu_Y, \max\{\sigma_X^2, \sigma_Y^2\}$

You do not need to worry whether $Z = X + Y$ is indeed Gaussian, this is given in the problem.

5. Let $X$ be a binomial random variable with parameters $p = 1/4$ and $n = 48$. Then it follows from the Chebyshev inequality that

- $P\{ 6 \leq X \leq 18 \} \leq \frac{3}{4}$
- $P\{ 7 \leq X \leq 17 \} \geq \frac{3}{4}$
- $P\{ X \leq 7 \text{ or } X \geq 17 \} \leq \frac{1}{4}$
- $P\{ X \leq 6 \text{ or } X \geq 18 \} \geq \frac{1}{4}$
Problem 3 (36 + 25 extra-credit points)

Alice, Bob, and Carol attend a dinner party together with \( n - 3 \) other people. Upon entering, all the diners hand over their business cards to the doorman for drawing in a lottery. There are two prizes in the lottery. Thus after the dinner is over, two cards are drawn at random (one after the other, without replacement) and their bearers each win a prize.

Express your answers to the questions below in terms of the number \( n \) of diners at the party. Show all your work.

a. What is the probability that Alice wins a prize?

\[
P(A) = \]
Next week Alice, Bob, and Carol attend this dinner party again. There are again \( n - 3 \) other people at this party, and exactly the same lottery is supposed to take place. However, this time Bob and Carol find out, just after the first card is drawn, that the doorman forgot to put their cards in the draw-basket. After much argument, the lottery organizers agree to put two cards each for Bob and Carol in the draw-basket before the second card is drawn.

b. What is the probability that Bob wins a prize?

\[
P(B) = \]

c. What is the probability that Alice wins a prize?

\[
P(A) = \]

Who has better chances of winning a prize: Alice or Bob?

\[ \square \text{Alice} \quad \square \text{Bob} \quad \square \text{same} \]
Extra-credit (25 points)

Just before Bob and Carol put two cards each in the draw-basket, a UCSD engineering student (who just took ECE109) walks up and says that she knows how to fix the problem in a jiffy. By making a simple modification of the lottery rules and procedures, that only takes seconds to implement, she equalizes all the diners’ probabilities of winning a prize. What modification did she propose?

State your answer precisely, and explain why it works!
Problem 4 (30 points)

Suppose that the height of the student population at UCSD is a Gaussian random variable $X$ with mean $\mu = 167$ and standard deviation $\sigma = 3$ (expressed in centimeters).

a. What percentage of students at UCSD are higher than 167cm?

$$ P\{X > 167\} = $$

b. What percentage of students at UCSD are higher than 170cm?

$$ P\{X > 170\} = $$

You may express your answer to this question in terms of the function $\Phi(\cdot)$, which is the cumulative distribution function (CDF) of the standard Gaussian random variable $N(0, 1)$. 
Now suppose that four UCSD students are chosen uniformly and independently at random. You may express your answers to the questions below in terms of the function $\Phi(\cdot)$, the CDF of $N(0, 1)$.

c. What is the probability that all four are higher than 170cm?

$$P\{\text{all four students are higher than 170cm}\} =$$

d. What is the probability that exactly two of the four students are higher than 170cm?

$$P\{\text{exactly two of the four students are higher than 170cm}\} =$$
Problem 5 (30 points)

In this problem, we consider two discrete random variables $X$ and $Y$ whose joint probability mass function is given by

$$p_{X,Y}(u, v) = \begin{cases} 
  c(2u + v) & u \in \{0, 1, 2\} \text{ and } v \in \{0, 1, 2\} \\
  0 & \text{otherwise}
\end{cases}$$

a. Compute the value of the constant $c$.

\[
c = \]

b. Compute $P\{X = 2 \cap Y = 1\}$ and $P\{X \geq 1 \mid Y \leq 1\}$

\[
P\{X = 2 \cap Y = 1\} = \quad P\{X \geq 1 \mid Y \leq 1\} = \]
c. Compute the marginal cumulative distribution functions (CDFs) of $X$ and $Y$.

$$F_X(u) = \begin{cases} \quad & \end{cases}$$

$$F_Y(v) = \begin{cases} \quad & \end{cases}$$
Problem 6 (28 points)

We have a rod of unit length. A point $X$ is chosen at random along the rod, and the rod is broken into two pieces at this point. Let $Y$ denote the length of the longer among the two pieces.

**Hint:** Assume that $X$ is a continuous random variable, distributed uniformly over the interval $(0, 1)$. Then express $Y$ in terms of $X$.

a. Compute the expectation and the variance of $Y$.

\[
E[Y] = \quad \text{Var}(Y) =
\]
b. Compute the cumulative distribution function (CDF) and the probability density function of $Y$. 

\[
F_Y(v) = \begin{cases} \\
\end{cases}
\]

\[
f_Y(v) = \begin{cases} \\
\end{cases}
\]
Problem 7 (40 points)

In this problem, we consider two jointly continuous random variables $X$ and $Y$ whose joint probability density function is given by

$$f_{X,Y}(u,v) = \begin{cases} 
  u + v & u \in [0,1] \text{ and } v \in [0,1] \\
  0 & \text{otherwise}
\end{cases}$$

a. Compute the marginal probability density functions of $X$ and $Y$.

$$f_X(u) = \begin{cases} 
  \text{[Blank]} & \text{[Blank]}
\end{cases}$$

$$f_Y(v) = \begin{cases} 
  \text{[Blank]} & \text{[Blank]}
\end{cases}$$

b. Are $X$ and $Y$ independent? Explain your answer and check the appropriate box below.

□ $X$ and $Y$ are independent  □ $X$ and $Y$ are not independent
c. Compute the covariance of \( X \) and \( Y \).

\[
\text{Cov}(X, Y) =
\]

d. Let \( Z = X^2 + Y^2 \). Compute the expected value of \( Z \).

\[
E[Z] =
\]

e. Compute \( P\{Y > X\} \).

\[
P\{Y > X\} =
\]