Question 1 (Nondeterministic complexity of inner product). Let $IP$ be the inner-product function over $n$-bit strings: $IP(x, y) = \langle x, y \rangle \mod 2$ where $x, y \in \{0, 1\}^n$.

(a) Prove that the co-nondeterministic communication complexity of $IP$, $N^0(IP) = \Theta(n)$.

(b) Prove that the nondeterministic communication complexity of $IP$, $N^1(IP) = \Theta(n)$.

Question 2 (Cover complexity vs communication complexity: A separation). We will work through an example showing that the theorem we saw in class, that $D(f) = O(N^0(f)N^1(f))$, cannot be significantly improved.

Let $n = m^2$ and interpret inputs $x, y \in \{0, 1\}^n$ as $m \times m$ binary matrices. Define a boolean function $f(x, y)$ as follows: $f(x, y) = 1$ if there exists $i \in [m]$ such that the $i$-th row of $x$ and the $i$-th row of $y$ are the same.

(a) Prove that the nondeterministic communication complexity of $f$, $N^1(f) = O(m)$.

(b) Prove that the co-nondeterministic communication complexity of $f$, $N^0(f) = O(m \log m)$.

(c) Prove that the deterministic communication complexity of $f$, $D(f) = \Omega(m^2)$.

Question 3 (Constant nondeterministic vs deterministic). Let $f(x, y)$ be a $n$-bit boolean function.

(a) Assume that $N^0(f) = O(1)$. Prove that $D(f) = O(1)$.

(b) In general, prove that $D(f) = F(N^0(f))$ for some function $F$. What is the best function you can attain? can you give an example showing it is tight?
**Question 4** (Rank+one sided cover implies a deterministic protocol). Let \( f : X \times Y \to \{0,1\} \), and let \( M_f \) be its corresponding communication matrix. Let \( r = \text{rank}(M_f) \) denote its rank over the reals. The goal is to prove that

\[
D(f) = O(N^0(f) \cdot \log r).
\]

(similarly, one can prove that \( D(f) = O(N^1(f) \cdot \log r) \)).

The following steps might be useful. Below, for a matrix \( M \) we denote \(|M|\) the number of elements in \( M \). We shorthand \( c = N^0(f) \).

(a) Let \( M \) be a sub-matrix of \( M_f \). Prove that \( M \) contains a monochromatic rectangle \( R \) of size \(|R| \geq \varepsilon|M|\) for \( \varepsilon = 2^{-O(c)} \).

(b) Write \( M \) as

\[
M = \begin{pmatrix}
R & A \\
B & C
\end{pmatrix}
\]

Prove that \( \text{rank}(A) + \text{rank}(B) \leq \text{rank}(M) + 1 \).

(c) Conclude that \( M \) can be partitioned either by rows, or by columns, into two matrices \( M_1, M_2 \) such that

(a) \( \text{rank}(M_1) \leq \text{rank}(M)/2 + O(1) \).

(b) \( |M_2| \leq (1 - \varepsilon)|M| \).

(d) Design a protocol tree computing \( f \) with \( r^{O(c)} \) leaves.

(e) Complete the proof.

(f) Does it matter if we compute the rank over the reals or over any other field?

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**Question 5.** Alice and Bob inputs are subspaces \( A, B \subset \mathbb{F}_2^n \). Define \( f(A, B) = 1 \) if \( A, B \) are orthogonal (namely \( \langle a, b \rangle = 0 \) for all \( a \in A, b \in B \)). Note that the inputs can be described using \( n^2 \) bits.

(a) Prove that the co-nondeterministic communication complexity of \( f \) is \( \Theta(n) \).

(b) Prove that the nondeterministic communication complexity of \( f \) is \( \Theta(n^2) \).