Kernel Properties:

Not all functions $K(x,z)$ are kernels.

Conditions for a function to be a kernel:

1. **Symmetry**: For all $x, z$, $K(x,z) = K(z,x)$
2. **Positive Semi Definiteness**: For a set of points $x^1, \ldots, x^m$, define kernel matrix as:

   $$K_{m \times m}, \ K_{ij} = K(x^i, x^j)$$

   For all $x^1, \ldots, x^m$, the kernel matrix is PSD.

   These are necessary and sufficient conditions.

How to show a function $K(x,z)$ is a kernel?

1. Either find a feature map $\phi$ s.t. $K(x,z) = \langle \phi(x), \phi(z) \rangle$
2. Or show conditions (1) and (2) hold (usually harder).

Example: Let $K(x,z) = \langle x, z \rangle^2$.

- (1) holds as $K(x,z) = K(z,x)$
- Let $t: m \times 1$ vector $= [t_1, \ldots, t_m]^T$, $x^1 \ldots x^m$ any $m$ vectors.

\[
t^TKt = \sum_{i=1}^m \sum_{j=1}^m t_i t_j K_{ij} = \sum_{i,j} t_i t_j \langle x^i, x^j \rangle^2
\]

Kernel matrix $K$

\[
K = \sum_{i,j} t_i t_j (\sum_{l=1}^d x^i_l x^j_l) = \sum_{i,j=1}^m t_i t_j \sum_{l,l'} x^i_l x^j_l x^{i'}_{l'} x^{j'}_{l'}
\]

For each pair $(l, l')$ we can take the summation over this pair out; for a fixed $(l, l')$ we sum over all $(i, j)$ pairs.
So we get:

\[
\sum_{l, l'} \sum_{i=1}^{m} \sum_{j=1}^{m} x_{le} x_{le'} x_{le'} x_{le'} t_i t_j
\]

\[
= \sum_{l, l'} (\sum_{i=1}^{m} x_{le} x_{le'} t_i) (\sum_{j=1}^{m} x_{le} x_{le'} t_j)
\]

\[
= \sum_{l, l'} (\sum_{i=1}^{m} x_{le} x_{le'} t_i)^2
\]  

The two parenthesized terms are the same, with different indices.

How to show a function \( K(x, z) \) is NOT a kernel?

Show a counter example to (1) or (2).

Example: \( K(x, z) = - \langle x, z \rangle \) is NOT a kernel. Why?

Pick \( x = \) any vector. Kernel matrix of \( x \) is a single scalar: \( x \neq 0 \)

\[
K = [\langle x, x \rangle] = [- \|x\|^2]
\]

For any \( K \) is not PSD. Why? For any \( 1 \times 1 \) \( t \), s.t. \( t \neq 0 \),

\[
t^TKt = - t^2 \|x\|^2 < 0
\]

So Condition (2) is violated.

Tip: If you suspect \( K(x, z) \) is not a kernel, try to find a small (1x1 or 2x2) matrix that is a counter example to (2).