Due Time : 12:00pm, Thursday February 7, 2019 Submit to Gradescope
The homework is about convex functions. For the 7 exercises, the first three exercises cover the
definition (3.2, 3.13, 3.14); the fourth verifies the convexity of a set of functions (3.16); the fifth
practices the operation of pointwise maximum and supremum (3.21); and the the last two work
on the conjugate function (3.36, 3.40). For the assignment, we first practice the derivation of the
gradient. We then work on a conjugate function with mathematical tools. The seven exercises
worth 1 point each and are graded by completion. The first assignments worth 6 points each
and the second worth 7 points.

I Exercises from textbook Chapter Three: 3.2, 3.13, 3.14, 3.16, 3.21, 3.36, 3.40

II Assignments
II.1 [Derivatives] Show the first and second order derivatives of function,
\[ f(x) = \frac{2x_1^3}{x_2} + x_1^2 + x_3, \quad x \in \mathbb{R}^3_+ \]
Is function \( f \) convex? Show your explanation.
\[
\nabla f(x) = \begin{pmatrix}
\frac{6x_1^2}{x_2} + x_3 \\
-\frac{2x_1^3}{x_2^2} + 2x_2 \\
x_1
\end{pmatrix}
\]
\[
\nabla^2 f(x) = \begin{pmatrix}
\frac{12x_1}{x_2} & -\frac{6x_1^2}{x_2^2} & 1 \\
-\frac{6x_1^2}{x_2^2} & \frac{4x_1^3}{x_2^3} + 2 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
is not positive semidefinite nor negative semidefinite. So \( f(x) \)
is not convex or concave.

II.2 [Conjugate Function] Derive the conjugate function \( f^*(y) \) of the following function,
\[ f(x) = (x_1 - 1)^2 + (x_2 - 3)^2, \quad x \in \mathbb{R}^2 \]
What does the curve of \( f^*(y) \) look like? Try to plot the figure with mathematical tools.

Let \( g(x, y) = y^T x - f(x) \) and the conjugate \( f^*(y) = \text{sup}_x g(x, y) \),
\[
\nabla_x g(x, y) = y - \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 3) \end{bmatrix}
\]
The function \( g(x, y) \) achieves maximum when \( \nabla_x g(x, y) = 0 \), so that
\[ x_1 = \frac{1}{2}y_1 + 1, \quad x_2 = \frac{1}{2}y_2 + 3 \]
The conjugate
\[ f^*(y) = \text{sup}_x g(x, y) = \frac{1}{4}y_1^2 + \frac{1}{4}y_2^2 + y_1 + 3y_2 \]
Figure 1. Plot of $f^*(y) = \frac{1}{3}y_1^2 + \frac{1}{3}y_2^2 + y_1 + 3y_2$. 