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Q1: Range an Null space

• The **range** of matrix $A \in \mathbb{R}^{m \times n}$ is $\mathcal{R}(A) = \{Ax \mid x \in \mathbb{R}^n\}$ or column subspace $col(A)$

• Row subspace $row(A) = \{A^T y \mid y \in \mathbb{R}^m\}$

• The **nullspace** of $A$ is defined to be $\mathcal{N}(A) = \{x \mid Ax = 0\}$
  • $\mathcal{N}(A) \perp row(A)$
  • $\dim(\mathcal{N}(A)) + \dim(row(A)) = n$
  • Analogy for column space

*How to find the range and null space of a matrix?*
Q1: Range an Null space

• Reduced Row Echelon Form
  • Determines whether $Ax = b$ is solvable and describes all the solutions
  • Gaussian elimination on rows
  • Pivot: the leading coefficient (first nonzero element from the left) is scaled to 1
  • Each column containing a leading 1 has zeros at other elements

Operation at column $k$
Q1: Range an Null space

• An example

\[
A_1 = \begin{pmatrix}
1 & 0 & 2 & 1 \\
1 & 1 & 5 & 2 \\
1 & 2 & 8 & 4
\end{pmatrix}
\]

At column 1, choose \(a_{11}\) as pivot

\[
\begin{align*}
R2\rightarrow R1, \\
R3\rightarrow R1
\end{align*}
\]

\[
A_1 \rightarrow A_2 = \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 3 & 1 \\
0 & 2 & 6 & 3
\end{pmatrix}
\]

At column 2, choose \(a_{32}\) as pivot

\[
\begin{align*}
R2\leftrightarrow R3, \\
R2/2 \\
R3-R2
\end{align*}
\]

\[
A_2 \rightarrow \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 2 & 6 & 3 \\
0 & 1 & 3 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 3 & 3/2 \\
0 & 1 & 3 & 1
\end{pmatrix} \rightarrow A_3 = \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 3 & 3/2 \\
0 & 0 & 0 & -1/2
\end{pmatrix}
\]
Q1: Range an Null space

No pivot at col 3. At column 4, choose \( a_{34} \) as pivot

\[
\begin{align*}
R3/-0.5 & \quad A_3 \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 3/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rightarrow \quad A_4 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
R1-R3, & \quad R2-R3*3/2 \quad \end{align*}
\]

• Columns 1, 2 and 4 are independent
  • The range \( A_1 x = x_1 \text{Col1} + \cdots + x_4 \text{Col4} \)

• \( \mathcal{R}(A_1) = \begin{Bmatrix} [1] \\ [1] \\ [2] \end{Bmatrix} \)

• The column spaces of \( A_1 \) and \( A_4 \) can be different, but their dimensions must be the same.
Q1: Range an Null space

• **Proposition:** given a matrix \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), after any sequence of elementary row operations \((A', b') = P(A, b)\), then solutions of \( Ax = b \) are the same as \( A'x = b' \).

• The null space of \( A \) is defined as \( \mathcal{N}(A) = \{ x \mid Ax = 0 \} \)
  
  • \( A_1x = 0 \iff A_4x = 0 \) where

\[
A_4 = \begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Solve

\[
x_1 + 2x_3 = 0 \\
x_2 + 3x_3 = 0 \\
x_4 = 0
\]

For example, a nonzero solution

\[
x = \begin{bmatrix}
-2 \\
-3 \\
1 \\
0
\end{bmatrix} \in \mathcal{N}(A)
\]

Which spans the null space because \( \dim(\mathcal{N}(A)) = 1 \).
Q1: Range an Null space

• What if there exist many variables?
  • The space of $x \in \mathcal{N}(A)$ could be format with the constrained and free variables.

$$A_4 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_1 + 2x_3 = 0$$
$$x_2 + 3x_3 = 0$$
$$x_4 = 0$$

$$x_1 = -2x_3$$
$$x_2 = -3x_3$$
$$x_3 = x_3$$
$$x_4 = 0$$

Free variable

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \text{ so } \mathcal{N}(A) = \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$
Q2: Eigenvalue Decomposition

• Let $A \in \mathbb{R}^{n \times n}$, $Ax = \lambda x$ leads to $\det(A - \lambda I) = 0$. Then it could be factorized as

$$A = Q\Lambda Q^{-1}$$

where $Q \in \mathbb{R}^{n \times n}$ contains the eigenvector $q_i$ of $A$, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ whose diagonal elements are the corresponding eigenvalues.

• The eigenvectors are usually normalized. If $A$ is symmetric then $Q$ is orthogonal.

• **Defective matrix**: cannot find $n$ independent eigenvectors, so it is not diagonalizable

$$Q = (q_1 \quad \cdots \quad q_n), \quad \Lambda = \begin{pmatrix} 
\lambda_1 & & & 0 \\
& \ddots & & \\
& & \lambda_r & \\
0 & & & 0 
\end{pmatrix}$$

Nil-potent
Q2: Eigenvalue Decomposition

- Could we tell the range and null space of matrix from its eigenvalue decomposition?

  \[ Ax = \lambda x \]

  - If \( \lambda_i \neq 0 \), then \( q_i \in \mathcal{R}(A) \)
  - If \( \lambda_i = 0 \), \( Aq_i = 0 \) then \( q_i \in \mathcal{N}(A) \)

Example:

\[
A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 1 & 2 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 0.1969 & 0.9123 & -0.5345 \\ 0.5154 & 0.3484 & -0.8018 \\ 0.8240 & -0.2154 & 0.2673 \end{pmatrix}, \Lambda = \begin{pmatrix} 9.4721 & 0.0000 \\ 0.0000 & 0.5279 \\ 0.0000 & 0.0000 \end{pmatrix}
\]
Q3: Singular Value Decomposition

• Let $A \in \mathbb{R}^{m \times n}$, $Ax = \lambda x$ leads to $\det(A - \lambda I) = 0$. Then it could be factorized as

$$A = U \Sigma V^*$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are unitary matrices which are orthogonal if $A$ is real. $\Sigma$ is a diagonal matrix with non-negative real diagonal, called singular values.

• The columns of $V$ are eigenvectors of $A^* A$

$$A^* A = V \Sigma (U^* U) \Sigma V^* = V \Sigma^2 V^*$$

• The columns of $U$ are eigenvectors of $AA^*$

$$AA^* = U \Sigma (V^* V) \Sigma U^* = U \Sigma^2 U^*$$

• The non-zero elements of $\Sigma$ are the square roots of the non-zero eigenvalues of $A^* A$ and $AA^*$

Related to eigenvalue decomposition, can we figure out the range and null space of a matrix from its SVD results?
Review: Dual Cone Properties

- Definition: Let $K$ be a cone, the set
  \[ K^* = \{ y \mid x^T y \geq 0 \text{ for all } x \in K \} \]
  is called the dual cone of $K$.
- A convex cone $K$ is **proper cone** if
  - $K$ is closed (contains its boundary)
  - $K$ is solid (has nonempty interior)
  - $K$ is pointed (contains no line)
- Let $K^*$ be the dual cone of a convex cone $K$. (see exercise 2.31)
  - $K^*$ is indeed a convex cone.
  - $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.
  - $K^*$ is closed.
  - The interior of $K^*$ is given by $\text{int} K^* = \{ y \mid y^T x > 0 \text{ for all } x \in \text{cl} K \}$.
  - If $K$ has nonempty interior then $K^*$ is pointed.
  - $K^{**}$ is the closure of $K$. (Hence if $K$ is closed, $K^{**} = K$.)
  - If the closure of $K$ is pointed then $K^*$ has nonempty interior.
Review: Dual Cone Properties

• How to prove that $K^*$ is indeed a convex cone?

See the definition of $K^*\{y \mid x^T y \geq 0 \text{ for all } x \in K\}$, $K^*$ is the intersection of a set of homogeneous halfspace ($x^T y \geq 0$) which include the origin as a boundary point. So $K^*$ is a closed convex cone.

• $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$

From the definition of $K^*$, if $y \in K_2^*$ then $x^T y \geq 0$ for all $x \in K_2$. So $x^T y \geq 0$ for all $x \in K_1$, $y$ is also in $K_1^*$.

• $K^{**}$ is the closure of $K$. (Hence if $K$ is closed, $K^{**} = K$.)

See the definition of $K^*$, $y \neq 0$ is the normal vector of a homogeneous halfspace containing $K$ iff $y \in K^*$. The intersection of all the homogeneous halfspaces containing a convex cone $K$ is the closure of $K$. Therefore the closure of $K$ is

$$\text{cl } K = \bigcap_{y \in K^*} \{x \mid y^T x \geq 0\} = \{x \mid y^T x \geq 0 \text{ for all } y \in K^*\} = K^{**}$$

• Try to prove the other properties.