Feedforward neural nets
The architecture

\[ y \]
\[ h^{(\ell)} \]
\[
\vdots
\]
\[ h^{(2)} \]
\[ h^{(1)} \]
\[ x \]
The value at a hidden unit

\[ h = \sigma(w_1 z_1 + w_2 z_2 + \cdots + w_m z_m + b) \]

\( \sigma(\cdot) \) is a nonlinear activation function, e.g. "rectified linear"

\( \sigma(u) = \begin{cases} u & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases} \)

How is \( h \) computed from \( z_1, \ldots, z_m \)?
The value at a hidden unit

How is $h$ computed from $z_1, \ldots, z_m$?

- $h = \sigma(w_1 z_1 + w_2 z_2 + \cdots + w_m z_m + b)$
- $\sigma(\cdot)$ is a nonlinear activation function, e.g. “rectified linear”

$$\sigma(u) = \begin{cases} u & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
Why do we need nonlinear activation functions?
The output layer

Classification with $k$ labels: want $k$ probabilities summing to 1.

\[ y_1 \quad y_2 \quad \cdots \quad y_k \]

\[ z_1 \quad z_2 \quad z_3 \quad \cdots \quad z_m \]
The output layer

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\[
y_1 \quad y_2 \quad \cdots \quad y_k
\]

\[
z_1 \quad z_2 \quad z_3 \quad \cdots \quad z_m
\]

- \( y_1, \ldots, y_k \) are linear functions of the parent nodes \( z_i \).
- Get probabilities using \textbf{softmax}:

\[
\Pr(\text{label } j) = \frac{e^{y_j}}{e^{y_1} + \cdots + e^{y_k}}.
\]
The complexity
The effect of depth

- **Universal approximator**
  Any function can be arbitrarily well approximated by a neural net with one hidden layer.
The effect of depth

- **Universal approximator**
  Any function can be arbitrarily well approximated by a neural net with one hidden layer.

- **Concerns about size**
  To fit certain classes of functions:
  - Either: one hidden layer of enormous size
  - Or: multiple hidden layers of moderate size
Learning a net: the loss function

Classification problem with $k$ labels.

- Parameters of entire net: $W$
- For any input $x$, net computes probabilities of labels:

  $$\text{Pr}_W(\text{label} = j|x)$$
Classification problem with $k$ labels.

- Parameters of entire net: $W$
- For any input $x$, net computes probabilities of labels:
  \[ \Pr_W(\text{label} = j|x) \]
- Given data set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$, loss function:
  \[ L(W) = -\sum_{i=1}^{n} \ln \Pr_W(y^{(i)}|x^{(i)}) \]
  (sometimes called **cross-entropy**).
Nature of the loss function

$L(w)$

$w$

$L(w)$

$w$
Variants of gradient descent

Initialize $W$ and then repeatedly update.

1. **Gradient descent**
   Each update involves the entire training set.

2. **Stochastic gradient descent**
   Each update involves a single data point.

3. **Mini-batch stochastic gradient descent**
   Each update involves a modest, fixed number of data points.
Derivative of the loss function

Update for a specific parameter: derivative of loss function wrt that parameter.
Chain rule

1. Suppose $h(x) = g(f(x))$, where $x \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$.

Then: $h'(x) = g'(f(x)) f'(x)$
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Then: $h'(x) = g'(f(x)) f'(x)$

2 Suppose $z$ is a function of $y$, which is a function of $x$.

Then:

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$
A single chain of nodes

A neural net with one node per hidden layer:

\[ x = h_0 \quad h_1 \quad h_2 \quad h_3 \quad \cdots \quad h_\ell \]

For a specific input \( x \),

- \( h_i = \sigma(w_i h_{i-1} + b_i) \)
- The loss \( L \) can be gleaned from \( h_\ell \)
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To compute \( \frac{dL}{dw_i} \) we just need \( \frac{dL}{dh_i} \):

\[
\frac{dL}{dw_i} = \frac{dL}{dh_i} \frac{dh_i}{dw_i} = \frac{dL}{dh_i} \sigma'(w_i h_{i-1} + b_i) h_{i-1}
\]
Backpropagation

- On a single forward pass, compute all the $h_i$.
- On a single backward pass, compute $dL/dh_\ell, \ldots, dL/dh_1$
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- On a single forward pass, compute all the $h_i$.
- On a single backward pass, compute $dL/dh_\ell, \ldots, dL/dh_1$

From $h_{i+1} = \sigma(w_{i+1}h_i + b_{i+1})$, we have

$$\frac{dL}{dh_i} = \frac{dL}{dh_{i+1}} \frac{dh_{i+1}}{dh_i} = \frac{dL}{dh_{i+1}} \sigma'(w_{i+1}h_i + b_{i+1}) w_{i+1}$$
Improving generalization

1. Early stopping
   - Validation set to better track error rate
   - Revert to earlier model when recent training hasn’t improved error
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   - Validation set to better track error rate
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2. Dropout
   During training, delete each hidden unit with probability 1/2, independently.

\[
\begin{align*}
y & \rightarrow h^{(\ell)} \\
& \rightarrow h^{(2)} \\
& \rightarrow h^{(1)} \\
& \rightarrow x
\end{align*}
\]