Today's learning goals

- Apply the Pumping Lemma in proofs of nonregularity
- Identify some nonregular sets
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$ such that:

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$.

If for every positive integer $p$, there exists a string $s : 1s12p$ then for any division of $s$ $S = xyz$ with $|y| > 0$ and $|xy| \leq p$. There exists an $i \geq 0$ such that $xy^iz \notin A$. $A$ is non-regular.
Proof strategy

To prove that a language L is **not** regular

- Consider arbitrary positive integer p.
- Prove that p isn't a pumping length for L.
- Conclude that L does not have any pumping length and is therefore **not** regular.
Another example

Claim: The set \( \{a^m b^m a^n \mid m, n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).

Consider the string

\[ s = \text{???} \]

1. \( |s| \geq p \) ?
2. \( s \) is in \( L \) ?
3. No matter how we cut \( s \) into three (viable) pieces, some related string obtained by "repeating" the middle part falls out of \( L \) ?
Aside...

To complete proofs with Pumping Lemma, we will need to build (useful) examples of strings with length $\geq p$ that are in a given language.

- $L_1 = \{a^m b^m a^n \mid m,n \geq 0\}$
- $L_2 = \{ww \mid w \text{ is a string over } \{0,1\}\}$
- $L_3 = \{ww^R \mid w \text{ is a string over } \{0,1\}\}$
Another example

Claim: The set \( \{a^n b^m a^n \mid m, n \geq 0\} \) is not regular.

Proof: … You must pick \( s \) carefully: we want \( |s| \geq p \) and \( s \) in \( L \) and \( s \) "can't be pumped”

Which choices of \( s \) cannot be used to complete the proof?
A. \( s = a^p b^p \)
B. \( s = aba \)
C. \( s = a^p b^p a^p \)
D. \( s = a^p b a^p \)
E. None of the above (all of these choices work).
Claim: \( A = \{a^m b^n a^m | n, m \geq 0 \} \) is non-regular.

**Proof:** Assume towards contradiction that \( A \) is regular.

\( \Rightarrow \) Pumping lemma \( \Rightarrow \) there is a pumping length \( p \).

Consider \( s = a^p b a^p \). So, if \( s = xyz \) with \( |xy| \leq p \),
\( x = a^r, y = a^k, z = a^t b a^p \) with \( r + k + t = p \),
\( k \geq 0. \) \( \boxed{xy^iz \in A} \)

Set \( i > 0 \). \( xz = a^r a^t b a^p \) and \( r + t < p \)
\( \Rightarrow xz \notin A \Rightarrow \) pumping lemma is not true \( \Rightarrow \) \( A \) is non-regular.
And another

Claim: The set \( \{ w w^R \mid w \text{ is a string over } \{0,1\} \} \) is not regular.

Proof: … You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \) and \( s \) "can't be pumped" … Consider \( i=\ldots\)

Which \( s \) and \( i \) let us complete the proof?
A. \( s = 0^p0^p, i=2 \)  B. \( s = 0110, i=0 \)  C. \( s = 0^p110^p, i=1 \)  D. \( s = 1^p001^p, i=3 \)  E. I don't know

\[ S=xy^2, \quad x=1^r, \quad y=1^k, \quad z=1^t001^p \]
How do we choose $i$?

Claim: The set $\{0^j1^k \mid j,k \geq 0 \text{ and } j \geq k \}$ is not regular.

Proof: … You must pick $s$ carefully: we want $|s| \geq p$ and $s$ in $L$ and $s$ "can't be pumped" … Consider $i = \ldots$

Which $s$ and $i$ let us complete the proof?
A. $s = 0^p1^p$, $i=2$  
B. $s = 0^p1^p$, $i=p$  
C. $s = 0^p1^p$, $i=1$  
D. $s = 0^p1^p$, $i=0$  
E. I don't know
Claim: The set
\[ L = \{ w \mid w \text{ has different #s of } 0\text{s and } 1\text{s OR has a } 1 \text{ before a } 0 \} \]
is nonregular.

Proof:
\[ L = L_1 \cup L_2 \]
\[ \overline{L} = \overline{L_1} \cap \overline{L_2} \]
where
\[ L_1 = \{ w \mid w \text{ has different # of } 0\text{s and } 1\text{s} \} \]
\[ L_2 = \{ w \mid w \text{ has a } 1 \text{ before a } 0 \} \]
if \( l_1, l_2 \) are regular then \( l_1 \cup l_2 \) is regular

if \( l_1 \) and \( l_2 \) are both non-regular, then \( l_1 \cup l_2 \) is non-regular.

if \( l_1 \cup l_2 \) is not regular then \( l_1 \) is not regular or \( l_2 \) is not regular.
Do we always need Pumping Lemma?

Claim: The set
\[ L = \{ w \mid w \text{ has different #s of 0s and 1s OR has a 1 before a 0} \} \]

is nonregular.

Proof: Suppose \( L \) is regular.

\[ L' = \{ w \in L \mid w \text{ has the same number of 0's and 1's AND w does not have a one before zero} \} \]

= \{ 0^n 1^n \mid n \geq 0 \}
Regular sets: not the end of the story

• Many **nice** / **simple** / **important** sets are not regular
• Limitation of the finite-state automaton model
  • Can't "count"
  • Can only remember finitely far into the past
  • Can't backtrack
  • Must make decisions in "real-time"
• We know computers are more powerful than this model…

*Which conditions should we relax?*
For next time

- Work on Individual HW3  **due Tuesday**

Pre class-reading for Wednesday: pages 111-112.

\[ \{0^n 1^n \mid n \geq 0\} \]