Today's learning goals

• Apply the Pumping Lemma in proofs of nonregularity
• Identify some nonregular sets
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = x y z$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^i z \in A$,
- $|xy| \leq p$. 

(Sipser p. 78 Theorem 1.70)
Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \). Consider the string \( s = 0^p1^p \). Then, \( s \) is in \( L \) and \( |s| = 2p \geq p \). Consider any division of \( s \) into three parts \( s = xyz \) with \( |y| > 0 \), \( |xy| \leq p \).

Since \( |xy| \leq p \), \( x = 0^k \), \( y = 0^m \), \( z = 0^r1^p \) with \( k + m + r = p \), and since \( |y| > 0 \), \( m > 0 \). Picking \( i = 0 \): \( xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p \), which is not in \( L \) because \( k + r < p \). Thus, no \( p \) can be a pumping length for \( L \) and \( L \) is not regular.
Proof strategy

To prove that a language $L$ is not regular

• Consider arbitrary positive integer $p$.
• Prove that $p$ isn't a pumping length for $L$.

• Conclude that $L$ does not have any pumping length and is therefore not regular.
Another example

Claim: The set \( \{a^m b^m a^n \mid m, n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).
Consider the string

\[ s = ??? \]

1. \( |s| \geq p \) ?
2. \( s \) is in \( L \) ?
3. No matter how we cut \( s \) into three (viable) pieces, some related string obtained by "repeating" the middle part falls out of \( L \) ?
To complete proofs with Pumping Lemma, we will need to
build (useful) examples of strings with \textbf{length} $\geq p$ that are
in a given language.

- $L_1 = \{a^m b^m a^n \mid m,n \geq 0\}$
- $L_2 = \{ ww \mid w \text{ is a string over } \{0,1\} \}$
- $L_3 = \{ ww^R \mid w \text{ is a string over } \{0,1\} \}$
Another example

Claim: The set \{a^m b^n a^n \mid m, n \geq 0\} is not regular.

Proof: \ldots You must pick s carefully: we want |s| \geq p and s in L and s "can't be pumped"

Which choices of s can be used to complete the proof?
A. s = a^p b^p  
B. s = aba  
C. s = a^p b^p a^p  
D. s = b^p  
E. None of the above
And another

**Claim:** The set \( \{w w^R \mid w \text{ is a string over } \{0,1\} \} \) is not regular.

**Proof:** … You must pick \( s \) carefully: we want \( |s| \geq p \) and \( s \) in \( L \) and \( s \) "can't be pumped" … **Consider** \( i=\ldots \)

Which \( s \) and \( i \) let us complete the proof?

A. \( s = 0^p0^p, i=2 \)  
B. \( s = 0110, i=0 \)  
C. \( s = 0^p110^p, i=1 \)  
D. \( s = 1^p001^p, i=3 \)  
E. I don't know
How do we choose i?

Claim: The set \( \{0^j1^k | j,k \geq 0 \text{ and } j \geq k \} \) is not regular.

Proof: … You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \) and \( s \) "can't be pumped" … Consider \( i = \ldots \)

Which \( s \) and \( i \) let us complete the proof?

A. \( s = 0^p1^p, \ i = 2 \)  B. \( s = 0^p1^p, \ i = p \)  C. \( s = 0^p1^p, \ i = 1 \)  D. \( s = 0^p1^p, \ i = 0 \)
E. I don't know
Do we always need Pumping Lemma?

Claim: The set
{w | w has different #s of 0s and 1s OR has a 1 before a 0}
is nonregular.

Proof:
Regular sets: not the end of the story

• Many **nice / simple / important** sets are not regular
• Limitation of the finite-state automaton model
  • Can't "count"
  • Can only remember finitely far into the past
  • Can't backtrack
  • Must make decisions in "real-time"
• We know computers are more powerful than this model…

*Which conditions should we relax?*
For next time

• Work on Individual HW3 due Tuesday

Pre class-reading for Wednesday: pages 111-112.