Today's learning goals

• Explain the limits of the class of regular languages
• Justify why the Pumping Lemma is true
• Apply the Pumping Lemma in proofs of nonregularity
• Identify some nonregular sets
Proving nonregularity

How can we prove that a set is non-regular?

A. Try to design a DFA that recognizes it and, if the first few attempts don't work, conclude there is none that does.
B. Prove that it's a strict subset of some regular set.
C. Prove that it's the union of two regular sets.
D. Prove that its complement is not regular.
E. I don't know.
Bounds on DFA

- in DFA, memory = states

- Automata can only "remember"…
  - …finitely far in the past
  - …finitely much information

- If a computation path visits the same state more than once, the machine can't tell the difference between the first time and future times it visited that state.
Example!

\{ 0^n1^n \mid n \geq 0 \}

What are some strings in this set?
What are some strings not in this set?

Compare to $L(0^*1^*)$
Design a DFA? NFA?
Example!

\{ 0^n1^n \mid n \geq 0 \}

What are some strings in this set?
What are some strings not in this set?

Compare to \( L(0^*1^*) \)
Design a DFA? NFA?
Pumping

- Focus on computation path through DFA
Pumping

- Focus on computation path through DFA
Pumping

• Focus on computation path through DFA

Idea: if one long string is accepted, then many other strings have to be accepted too
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = x\ y\ z$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$. 
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xy^i z$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^i z \in A$,
- $|xy| \leq p$. 

Sipser p. 78 Theorem 1.70
Pumping Lemma
Pumping Lemma

• True for **all** (but not only) regular sets.
  
  • Can't be used to prove that a set *is* regular
  • Can be used to a prove that a set *is not* regular … **how?**
Negation

flash-back to CSE 20 😊

• Pumping lemma "There is \( p \), where \( p \) is a pumping length for \( L \)"

• Given a specific number \( p \), it being a pumping length for \( L \) means

\[
\forall s \left( (|s| \geq p \land s \in L) \rightarrow \exists x \exists y \exists z \left( s = xyz \land |y| > 0 \land |xy| \leq p \land \forall i (xy^iz \in L) \right) \right)
\]

• So \( p \) not being a pumping length of \( L \) means

\[
\exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L) \right) \right)
\]
Proof strategy

To prove that a language L is not regular

• Consider arbitrary positive integer p.
• Prove that p isn't a pumping length for L.

• Conclude that L does not have any pumping length and is therefore not regular.
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \).

How? Want to show that there is some string that *should* be pump'able but isn't.
Using the Pumping Lemma

\( L = \{0^n1^n \mid n \geq 0\} \) **CLAIM:** \( p \) is not a pumping length for \( L \).

**How would you prove the claim?**

A. Find a string with length \( \geq p \) that is not in \( L \).
B. Find a string with length \( < p \) that is in \( L \).
C. None of the above.

\( \exists s \ (|s| \geq p \land s \in L \land \forall x \forall y \forall z ((s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i(xy^iz \notin L)) \)
Using the Pumping Lemma

\[ L = \{0^n1^n \mid n \geq 0\} \]  

**CLAIM:** \( p \) is not a pumping length for \( L \).

**WTS**

\[ \exists s \left( |s| \geq p \land s \in L \land \forall x \forall y \forall z \left( (s = xyz \land |y| > 0 \land |xy| \leq p) \rightarrow \exists i (xy^iz \notin L) \right) \right) \]

Find a string \( s \) such that

1. \( |s| \geq p \)
2. \( s \) is in \( L \)
3. No matter how we cut \( s \) into three (viable) pieces, some related string obtained by repeating the middle part falls out of \( L \).
Using the Pumping Lemma

**Claim:** The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

**Proof:** Consider an arbitrary positive integer. WTS $p$ is not a pumping length for $L$.

Consider the string 

$$s = 0^p1^p.$$ 

1. $|s| \geq p$ ?
2. $s$ is in $L$ ?
3. No matter how we cut $s$ into three (viable) pieces, some related string obtained by repeating the middle part falls out of $L$ ?
Using the Pumping Lemma

**Claim:** The set \( L = \{0^n1^n | n \geq 0\} \) is not regular.

**Proof:** Consider an arbitrary positive integer. WTS \( p \) is not a pumping length for \( L \). Consider the string \( s = 0^p1^p \). Then, \( s \) is in \( L \) and \( |s| = 2p \geq p \). Consider any division of \( s \) into three parts 

\[ s = xyz \text{ with } |y| > 0, \ |xy| \leq p. \]

Since \( |xy| \leq p \), \( x = 0^k \), \( y = 0^m \), \( z = 0^r1^p \) with \( k+m+r = p \),

and since \( |y| > 0, m > 0 \). Picking \( i = 0 \): \( xy^iz = xz = 0^k0^r1^p = 0^{k+r}1^p \),

which is not in \( L \) because \( k+r < p \). Thus, no \( p \) can be a pumping length for \( L \) and \( L \) is not regular.
WOW!
For next time

• Work on Group Homework 2 due Saturday

Pre class-reading for Monday: page 77.