Today's learning goals

• Distinguish between a theorem, an axiom, lemma, a corollary, and a conjecture.
• Recognize direct proofs
• Recognize proofs by contraposition
• Recognize proofs by contradiction
• Recognize fallacious “proofs”
• Evaluate which proof technique(s) is appropriate for a given proposition
  • Direct proof
  • Proofs by contraposition
  • Proofs by contradiction
  • Proof by cases
  • Constructive existence proofs
• Correctly prove statements using appropriate style conventions, guiding text, notation, and terminology

Textbook references: Sections 1.7-1.8 (with reference to 1.6)
Review: quantified statements

In which domain is this statement not true?

A. All real numbers in the closed interval \([0,1]\).
B. The set of integers \(\{1,2,3\}\).
C. All real numbers.
D. All positive real numbers.
E. All positive integers.

\[ \forall x \forall y \exists z (xy = z) \]
And in the other direction . . .  

• "The product of two negative real numbers is positive."
And in the other direction ...  

- "The difference between a real number and itself is zero."
And in the other direction … Rosen p. 66 #23

• "A negative real number does not have a square root that is a real number."
And in the other direction ...  

- "Every positive real number has exactly two square roots."
"A proof is a valid argument that establishes the truth of a statement"
For the sake of argument

"A proof is a valid argument that establishes the truth of a statement"

Valid: truth of the conclusion (final statement) is guaranteed by the truth of the premises (preceding statements)

Argument: sequence of statements ending with a conclusion
For the sake of argument

Axioms: statements assumed to be true

Rules of inference + definitions

Theorem
Proposition
Fact
Result
Lemma
Corollary
What rules of inference are allowed?  

Argument form is valid if no matter which propositions are substituted in, the conclusion is true if all the premises are true.
Fancy names

- Modus ponens
- Modus tollens
- Hypothetical syllogism …
- Universal instantiation
- Universal generalization
- Existential instantiation
- Existential generalization

Rosen 1.6
Translating to proof strategy

• Modus ponens

Direct proof

To prove that "If P, then Q"
- Assume P is true.
- Use rules of inference, axioms, definitions to...
- … conclude Q is true.
Translating to proof strategy

- Modus tollens

Proof by contraposition

To prove that "If P, then Q"
- Assume Q is false.
- Use rules of inference, axioms, definitions to...
- ... conclude P is also false.

Both modus ponens and modus tollens apply when proving a conditional statement.
What's the main logical structure in the following sentence?

\[ n^2 + 1 \geq 2^n \text{ when } n \text{ is a positive integer with } 1 \leq n \leq 4 \]

A. Conditional (p → q)
B. Universal statement
C. Conjunction ("AND")
D. Inequality (≥)
E. What does this question even mean?
Are these statements true?

I. $n^2 + 1 \geq 2^n$ when $n$ is a positive integer with $1 \leq n \leq 4$
II. $n^2 + 1 \geq 2^n$ when $n$ is a positive integer

A. Both are true.
B. Both are false.
C. I is true and II is false.
D. II is true and I is false.
E. I don't know how to evaluate the truth value of these sentences.
Universal conditionals

\[ n^2 + 1 \geq 2^n \text{ when } n \text{ is a positive integer with } 1 \leq n \leq 4 \]
\[ n^2 + 1 \geq 2^n \text{ when } n \text{ is a positive integer} \]

can be translated to

\[ \forall n \ (\ldots \rightarrow \ldots) \]
Universal conditionals

\[ \forall n \ (\ldots \rightarrow \ldots) \]

To prove this statement is \textbf{false}:

To prove this statement is \textbf{true}:
Universal conditionals

\[ \forall n \ (\ldots \rightarrow \ldots) \]

To prove this statement is **false**:
Find a **counterexample**.

To prove this statement is **true**:
Select a **general element** of the domain, use rules of inference (e.g. direct proof, proof by contrapositive, etc.) to prove that the conditional statement is true of this element, c conclude that it holds of all members of the domain.
Universal conditionals

Claim: \( n^2 + 1 \geq 2^n \) when \( n \) is a positive integer is false.

Proof:
**Universal conditionals**

**Claim:** \( n^2 + 1 \geq 2^n \) when \( n \) is a positive integer is false.

**Proof:** Since this is a **universal** conditional statement, to prove that it is false, it's enough to find one counterexample. That is, we need a specific value of \( n \) which is a positive integer and for which

A. \( n^2 + 1 > 2^n \)
B. \( n^2 + 1 < 2^n \)
C. \( n^2 + 1 \leq 2^n \)
D. \( n^2 + 1 \geq 2^n \)
E. What's a universal conditional?
Universal conditionals

Claim: \( n^2 + 1 \geq 2^n \) when \( n \) is a positive integer is false.

Proof: Since this is a universal conditional statement, to prove that it is false, it's enough to find one counterexample. That is, we need a specific value of \( n \) which is a positive integer and for which \( n^2 + 1 < 2^n \). Consider ....

A. \( n = 0 \)
B. \( n = 1 \)
C. \( n = 4 \)
D. \( n = 5 \)
E. None of the above
Universal conditionals

Claim: \( n^2 + 1 \geq 2^n \) when \( n \) is a positive integer is false.

Proof: Since this is a universal conditional statement, it's enough to find one counterexample. That is, we need a specific value of \( n \) which is a positive integer and for which \( n^2 + 1 < 2^n \).

Consider \( n = 5 \). Then, for this example, the LHS of the inequality evaluates to \( 5^2 + 1 = 26 \) and the RHS evaluates to \( 2^5 = 32 \).

Therefore, LHS < RHS, contradicting the statement. Thus, there is (at least) one counterexample to the universal statement and hence the universal statement is false.
Universal conditionals

Claim: \( n^2 + 1 \geq 2^n \) when \( n \) is a positive integer with \( 1 \leq n \leq 4 \) is true.

Proof: WTS \( \forall n \ (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n) \)

Let \( n \) be a positive integer (a general element of the domain). For a direct proof, assume

A. \( n \geq 1 \)
B. \( n^2 + 1 \geq 2^n \)
C. \( n \leq 4 \)
D. \( n^2 + 1 < 2^n \)
E. None of the above
Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when $n$ is a positive integer with $1 \leq n \leq 4$ is true.

Proof: WTS $\forall n \ (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)$

Let $n$ be a positive integer (a general element of the domain). For a direct proof, assume that $1 \leq n \leq 4$.

Goal: WTS that $n^2 + 1 \geq 2^n$

How can we use the hypothesis?
Universal conditionals

Claim: $n^2 + 1 \geq 2^n$ when $n$ is a positive integer with $1 \leq n \leq 4$ is true.

Proof: WTS $\forall n \ (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)$

Let $n$ be a positive integer (a general element of the domain). For a direct proof, assume that $1 \leq n \leq 4$.

Goal: WTS that $n^2 + 1 \geq 2^n$

Since we assume that $n$ is an integer between 1 and 4, there are only four possible cases. We check that the conclusion is true in each one.
Universal conditionals

**Claim:** $n^2 + 1 \geq 2^n$ when $n$ is a positive integer with $1 \leq n \leq 4$ is true.

**Proof:** WTS \quad \forall n \ (1 \leq n \leq 4 \rightarrow n^2 + 1 \geq 2^n)

Let $n$ be a positive integer (a general element of the domain). For a direct proof, assume that $1 \leq n \leq 4$.

**Goal:** WTS that $n^2 + 1 \geq 2^n$

**Case 1:** Assume $n=1$. Then LHS is $1^2+1 = 2$; RHS is $2^1=2$ so LHS $\geq$ RHS. 😊

**Case 2:** Assume $n=2$. Then LHS is $2^2+1 = 5$; RHS is $2^2=4$ so LHS $\geq$ RHS. 😊

**Case 3:** Assume $n=3$. Then LHS is $3^2+1 = 10$; RHS is $2^3=8$ so LHS $\geq$ RHS. 😊

**Case 4:** Assume $n=4$. Then LHS is $4^2+1 = 17$; RHS is $2^4=16$ so LHS $\geq$ RHS. 😊
Universal conditionals

Claim: \( n^2 + 1 \geq 2^n \) when \( n \) is a positive integer.

Proof: WTS \( \forall n \geq 1 \) \( n^2 + 1 \geq 2^n \)
Let \( n \) be a positive integer (a general element of the domain). For a direct proof, assume that \( 1 \leq n \leq 4 \).

Goal: WTS that \( n^2 + 1 \geq 2^n \)

Case 1: Assume \( n=1 \). Then LHS is \( 1^2+1 = 2 \); RHS is \( 2^1=2 \) so LHS \( \geq \) RHS. 😊

Case 2: Assume \( n=2 \). Then LHS is \( 2^2+1 = 5 \); RHS is \( 2^2=4 \) so LHS \( \geq \) RHS. 😊

Case 3: Assume \( n=3 \). Then LHS is \( 3^2+1 = 10 \); RHS is \( 2^3=8 \) so LHS \( \geq \) RHS. 😊

Case 4: Assume \( n=4 \). Then LHS is \( 4^2+1 = 17 \); RHS is \( 2^4=16 \) so LHS \( \geq \) RHS. 😊

Exhaustive proof (cf. page 93)
Reminder: perfect squares

An integer $a$ is a perfect square iff there is some integer $b$ such that

$$a = b^2$$

Which of the following is not a perfect square?

A. 0
B. 1
C. 2
D. 4
E. More than one of the above
An integer $a$ is a perfect square iff there is some integer $b$ such that:

$$a = b^2$$

Which of the following is not true?

A. If $n$ is a perfect square, so is $n^2$.
B. If $n^2$ is a perfect square, so is $n$.
C. If $n^2$ is not a perfect square, neither is $n$.
D. For each $n$, $n^2$ is a perfect square.
E. More than one of the above.
Reminder: perfect squares

Theorem: For integer k>1, then \(2^k-1\) is not a perfect square.

Proof: Let k be an integer.

What would we assume in a direct proof?
A. \(k > 1\)
B. \(k \leq 1\)
C. \(2^k-1\) is not a perfect square
D. \(2^k-1\) is a perfect square
E. More than one of the above
Reminder: perfect squares

Theorem: For integer $k > 1$, then $2^k - 1$ is not a perfect square.

Proof: Let $k$ be an integer.

What would we assume in a proof by contraposition?
A. $k > 1$
B. $k \leq 1$
C. $2^k - 1$ is not a perfect square
D. $2^k - 1$ is a perfect square
E. More than one of the above
Reminder: perfect squares

Theorem: For integer $k>1$, then $2^k-1$ is not a perfect square.

Proof: ???
Tangent, for a moment
Reminder: evens and odds

An integer $a$ is even iff there is some integer $b$ such that

$$a = 2b$$

Which of the following is equivalent to definition of $a$ being even?

A. $a/2$
B. $a \div 2$ is an integer.
C. $a \mod 2$ is zero.
D. $2a$ is an integer.
E. More than one of the above.
Reminder: divisibility

For integers $a, b$ with $a$ nonzero

$$\exists c (b = ac)$$

means

$$a | b$$, a.k.a.

$$\frac{b}{a} \in \mathbb{Z}$$

"a divides b"  "b is an integer multiple of a"
Reminder: evens and odds

An integer \( a \) is **even** if there is some integer \( b \) such that

\[
a = 2b
\]

Which of the following numbers is **not** even?

A. 0, even though \( 0 = 2(0) \)
B. 0.5, even though \( 0.5 = 2 (0.25) \)
C. -2, even though \( -2 = 2 (-1) \)
D. I don't know.
Reminder: evens and odds

An integer $a$ is **even** if there is some integer $b$ such that

$$a = 2b.$$ 

An integer $a$ is **odd** iff

- it's not even
- there is some integer $b$ such that

$$a = 2b + 1.$$
Flexing proof muscles

**Theorem:** If \( n \) is even, then so is \( n^2 \).

**Proof:**
Flexing proof muscles

Theorem: If $n$ is odd, then so is $n^2$.

Proof:
Flexing proof muscles

Theorem: If $n^2$ is even, then so is $n$.

Proof:
Flexing proof muscles

**Theorem:** If $n^2$ is odd, then so is $n$.

**Proof:**
Theorem: For integers $k > 1$, then $2^k - 1$ is not a perfect square.

Proof: ???
Theorem: For integers \( k > 1 \), then \( 2^k - 1 \) is not a perfect square.

Proof: Try proof by contraposition again… Let \( k \) be an integer.

Assume: \( 2^k - 1 \) is a perfect square

Note: \( k \) must be nonnegative because perfect squares are integers.

WTS: \( k \leq 1 \).

Note: using above, what we want to show is that \( k = 0 \) or 1.
Theorem: For integers \( k > 1 \), then \( 2^k - 1 \) is not a perfect square.

Proof: Try proof by contraposition again… Let \( k \) be an integer.

Assume: \( 2^k - 1 \) is a perfect square

Note: \( k \) must be nonnegative because perfect squares are integers.

By definition of perfect square, there is \( c \) such that \( 2^k - 1 = c^2 \).

Either \( k = 0 \) (i.e. \( c = 0 \)) or \( 2^k - 1 \) is a positive integer.

Is this integer even or odd?
Overall strategy

• Do you believe the statement?
  • Try some small examples.
• Determine logical structure + main connective.
• Determine relevant definitions.
• Map out possible proof strategies.
  • For each strategy: what can we assume, what is the goal?
  • Start with simplest, move to more complicated if/when get stuck.
Proof by contradiction

Idea: To prove $P$, instead, we prove that the conditional

$$(-P) \rightarrow F$$

is true. But, the only way for a conditional to be true if its conclusion is false is for its hypothesis to be false too.

Conclude: $(-P)$ is false, i.e. $P$ is true!
Next up

• Using proofs in the context of sets