Today's learning goals

• Define and compute the cardinality of a set: Finite sets, countable sets, uncountable sets
• Use functions to compare the sizes of sets
• Determine and prove whether a given binary relation is an equivalence relation using the defining properties
  • symmetry
  • reflexivity
  • transitivity
• Determine and prove whether a given binary relation is an equivalence relation using the associated partition.
Cardinality

- Finite sets
  \[ |A| = n \text{ for some nonnegative int } n \]

- Countably infinite sets
  \[ |A| = |\mathbb{Z}^+| \text{ (informally, can be listed out)} \]

- Uncountable sets
  Infinite but not in bijection with \( \mathbb{N} \)
Cardinality

- Finite sets

\[ |A| = n \text{ for some nonnegative int } n \]

Which of the following sets is **not** finite?

A. \( \emptyset \)
B. \([0, 1]\)
C. \( \{x \in \mathbb{Z} | x^2 = 1\} \)
D. \( \mathcal{P}\{1, 2, 3\} \)
E. None of the above (they're all finite)
Cardinality

• Countable sets \( A \) is finite or \(|A| = |\mathbb{Z}^+|\) (informally, can be listed out)

Examples: \( \emptyset \), \( \{x \in \mathbb{Z} | x^2 = 1\} \), \( \mathcal{P}(\{1, 2, 3\}) \), \( \mathbb{Z}^+ \)

and also …
- the set of odd positive integers
- the set of all integers
- the set of positive rationals
- the set of negative rationals
- the set of rationals

Example 1
Example 3
Example 4
Cardinality and subsets, redux

Which of the following is not true?

A. If A and B are both countable then AUB is countable.
B. If A and B are both countable then A\cap B is countable.
C. If A and B are both countable then AxB is countable.
D. If A is countable then P(A) is countable.
E. None of the above
There is an uncountable set! \textit{Rosen example 5, page 173-174}

Cantor's diagonalization argument

Theorem: For every set $A$, $|A| \neq |\mathcal{P}(A)|$
There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

**Theorem:** For every set $A$, $|A| \neq |\mathcal{P}(A)|$

An example to see what is necessary. Consider $A = \{a,b,c\}$. What would we need to prove that $|A| = |\mathcal{P}(A)|$?
Cantor's diagonalization argument

**Theorem:** For every set $A$, $|A| 
eq |\mathcal{P}(A)|$

**Proof:** (Proof by contradiction)

Assume towards a contradiction that $|A| = |\mathcal{P}(A)|$. By definition, that means there is a **bijection** $A \rightarrow \mathcal{P}(A)$. 

There is an uncountable set!  
*Rosen example 5, page 173-174*
There is an uncountable set!  
Rosen example 5, page 173-174

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

\[ a \in D \text{ iff } a \notin f(a) \]
Cantor's diagonalization argument

Consider the subset $D$ of $A$ defined by, for each $a$ in $A$:

$$a \in D \iff a \notin f(a)$$

Define $d$ to be the pre-image of $D$ in $A$ under $f$ $f(d) = D$

Is $d$ in $D$?

- If yes, then by definition of $D$, $d \notin f(d) = D$ a contradiction!
- Else, by definition of $D$, $\neg(d \notin f(d))$ so $d \in f(D) = D$ a contradiction!
Cardinality

- Uncountable sets Infinite but not in bijection with $\mathbb{Z}^+$

*Examples*: the power set of any countably infinite set

*and also* …

- the set of **real** numbers
  - $(0,1)$
  - $(0,1]$
Cardinality and subsets

Suppose $A$ and $B$ are sets and $A \subseteq B$.

A. If $A$ is finite then $B$ is finite.
B. If $A$ is countable then $B$ is uncountable.
C. If $B$ is infinite then $A$ is finite.
D. If $B$ is uncountable then $A$ is uncountable.
E. None of the above.
Size as a relation

- Cardinality lets us compare and group sets.

A is related to B iff $|A| = |B|$
Size as a relation

- Cardinality lets us compare and group sets.

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