Today's learning goals

• Explain the steps in a proof by (strong) mathematical induction
• Use (strong) mathematical induction to prove
  • correctness of identities and inequalities
  • properties of algorithms
  • properties of geometric constructions
• Represent functions in multiple ways
• Use functions to define sequences: arithmetic progressions, geometric progressions
• Use and prove properties of recursively defined functions and recurrence relations (using induction)
• Use and interpret Sigma notation
**Nim**

Two players take turns removing any positive # of jellybeans at a time from one of two piles in front of them. The piles start out with equal #s. The player who removes the last jellybean wins the game.

A. The first player has a strategy to always win.
B. The second player has a strategy to always win.
C. One of the players has a strategy to always win, but which player depends on how many jellybeans there are.
D. Who wins is random.
E. None of the above.
Nim

Two players take turns removing any positive # of jellybeans at a time from one of two piles in front of them. The piles start out with equal #s. The player who removes the last jellybean wins the game.

n=1  Who wins?
Nim

Two players take turns removing any positive # of jellybeans at a time from one of two piles in front of them. The piles start out with equal #s. The player who removes the last jellybean wins the game.

n=2   Who wins?
Nim

Two players take turns removing any positive # of jellybeans at a time from one of two piles in front of them. The piles start out with equal #s. The player who removes the last jellybean wins the game.

Idea: 2nd player takes the same amount 1st player took but from opposite pile. ...Game reduces to same setup but with fewer jellybeans.
Strong induction

To show that some statement $P(n)$ is true about all positive integers $n$,

1. Verify that $P(1)$ is true.
2. Let $k$ be an arbitrary positive integer. Show that

$$[P(1) \land P(2) \land \cdots \land P(k)] \rightarrow P(k + 1)$$

is true.
Nim

Two players take turns removing any positive # of jellybeans at a time from one of two piles in front of them. The piles start out with equal #s. The player who removes the last jellybean wins the game.

Theorem: the second player can always guarantee a win.

Proof: By Strong Mathematical Induction, on # jellybeans in each pile.

1. **Basis step** WTS if piles each have 1, then 2nd player can win.

2. **Strong Induction hypothesis** Let k be a positive integer. Assume that 2nd player can win whenever there are j jellybeans in each pile, for each j between 1 and k (inclusive).

3. **Induction step** WTS 2nd player has winning strategy when start with k+1 jellybeans in each pile.
Fibonacci numbers

\[ f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \]

What are some sample values?

How quickly do these values grow?
Fibonacci numbers

\[ f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \]

Theorem: For each integer \( n \geq 2 \), \( f_n \geq 1.5^{n-2} \)

Proof: By Strong Mathematical Induction, on \( n \geq 2 \).

1. **Basis step** WTS \( f_2 \geq 1.5^{2-2} \).
2. **Strong Induction hypothesis** Let \( k \) be an integer, \( k \geq 2 \). Assume inequality is true for each integer \( j \), \( 2 \leq j \leq k \).
3. **Induction step** WTS statement is true about \( f_{k+1} \).
Fibonacci numbers

\( f_0 = 1, \ f_1 = 1, \ f_n = f_{n-1} + f_{n-2} \)

1. **Basis step** \( \text{WTS } f_2 \geq 1.5^{2-2} \).

LHS = \( f_2 = 1 + 1 = 2 \).
RHS = \( 1.5^{2-2} = 1.5^0 = 1 \).
Since \( 2 > 1 \), LHS > RHS so, in particular, LHS \( \geq \) RHS 😊
Fibonacci numbers

\[ f_0 = 1, \ f_1 = 1, \ f_n = f_{n-1} + f_{n-2} \]

**Induction step** Let \( k \) be an integer with \( k \geq 2 \).
Assume as the **strong induction hypothesis** that
\[ f_j \geq 1.5^{j-2} \]
for each integer \( j \) with \( 2 \leq j \leq k \).
**WTS** that \( f_{k+1} \geq 1.5^{(k+1)-2} \)
By definition of Fibonacci numbers, since \( k+1 > 1 \), \( f_{k+1} = f_k + f_{k-1} \).
Therefore, LHS = \( f_{k+1} = f_k + f_{k-1} \).

*Idea:* apply **strong induction hypothesis** to \( k \) and \( k-1 \). Can we do it?
Fibonacci numbers

Case 1: $k=2$ and WTS that $f_3 \geq 1.5^{(3)-2}$

Case 2: $k>2$ and WTS that $f_{k+1} \geq 1.5^{(k+1)-2}$ and strong IH applies to $k$, $k-1$ (because both $k$, $k-1$ are greater than or equal to 2 and less than $k+1$).

So let's prove each of these cases in turn:

Case 1: $k=2$ and WTS that $f_3 \geq 1.5^{(3)-2}$

By definition of Fibonacci numbers, $LHS = f_3 = f_2 + f_1 = 2 + 1 = 3$.

By algebra, $RHS = 1.5^{3-2} = 1.5$  Since $3 > 1.5$, $LHS > RHS$ 😊
Case 2: \( k > 2 \) and WTS that \( f_{k+1} \geq 1.5^{(k+1)-2} \) and strong IH applies to \( k, k-1 \) (because both \( k, k-1 \) are greater than or equal to 2 and less than \( k+1 \)).

\[
LHS = f_{k+1} = f_k + f_{k-1} \geq 1.5^{k-2} + 1.5^{(k-1)-2} = 1.5^{k-3}(1.5+1) = 1.5^{k-3}(2.5) > 1.5^{k-3}(2.25) = 1.5^{k-3}1.5^2 = 1.5^{k-1} = 1.5^{(k+1)-2} = RHS.
\]
Flavors of induction

- Mathematical induction
- Structural induction
- Strong induction
Fulfilling promises

- We now have all the tools we need to rigorously prove
  - Correctness of greedy change-making algorithm with quarters, dimes, nickels, and pennies *Proof by contradiction, Rosen p. 199*
  - The division algorithm is correct *Strong induction, Rosen p. 341*
  - Russian peasant multiplication is correct *Induction*
  - Largest n-bit binary number is $2^n-1$ *Induction, Rosen p. 318*
  - Correctness of base b conversion (Algorithm 1 of 4.2), *Strong induction*
  - Size of the power set of a finite set with n elements is $2^n$ *Induction, Rosen p. 323*
  - Any int greater than 1 can be written as product of primes *Strong induction, Rosen p. 323*
  - There are infinitely many primes *Proof by contradiction, Rosen p. 260*
  - Sum of geometric progressions \[ \sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1} \] when r≠1, *Induction, Rosen p. 318*
Cautionary tales

• The **basis step** is absolutely necessary … and might need more than one!

• Make sure to stay in the **domain**.

  *Recommended practice*
  Section 5.1 #49, 50, 51
  Section 5.2 #32

• A few **examples** do not guarantee a pattern:
cake cutting conundrum. Join all pairs of points among \( N \) marked on circumference of cake.
Reminders

• Exam 2 is next class: **Tuesday Feb 28**
  • One note card allowed.
  • Seat map posted on Piazza.

• Review sessions this weekend.
  • Sat, Feb 25, 11:00am – 12:50pm PETERSON 108
  • Mon, February 27, 8:00pm – 9:50pm PETERSON 110

• Extra office hours available.
• Practice exam available on Piazza.