CSE 20
DISCRETE MATH

Winter 2017

http://cseweb.ucsd.edu/classes/wi17/cse20-ab/
Today's learning goals

• Explain the steps in a proof by mathematical and/or structural induction
• Use mathematical induction to prove identities about sizes of finite sets
• Use a recursive definition of a set to determine the elements of the set
• Write a recursive definition of a given set
• Use structural induction to prove properties of a recursively defined set
• Define functions from $\mathbb{N}$ to $\mathbb{R}$ using recursive definitions
Mathematical induction

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it’s true about 0 i.e. $P(0)$
2. Show $\forall k \ ( P(k) \rightarrow P(k + 1) )$ Hence conclude $P(1),...$
Sizes of (finite) sets

If $S$ is a set with exactly $n$ distinct elements, with $n$ a nonnegative integer, then $S$ is finite set and $|S| = n$.

$|\emptyset| = 0$
$|\{1\}| = 1$
$|\{1, 1, 2, 3, 8/4\}| = 3$
Operations on sets

If the sets $A$, $B$ are finite then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Two sets $A$ and $B$ are **disjoint** iff

$$A \cap B = \emptyset$$

equivalently, $A$ and $B$ are disjoint iff $|A \cup B| = |A| + |B|$
Operations on sets

If the sets $A$, $B$ are finite then

$$|A \times B| = |A| \cdot |B|$$

$|B|$ many elements for each of the $|A|$ many elements in $A$
How do we prove this?

**Theorem:** For any nonnegative integer n, if A is any set of size n, then for any finite set B, $|A \times B| = |A||B|$.

**Proof by Mathematical Induction:**

1. **Basis step** WTS if A is empty, then for any finite set B, $|A \times B| = |A||B| = 0$.
2. **Inductive hypothesis** Let k be a nonnegative integer. Assume that for any set C of size k and any finite set D, $|C \times D| = |C||D|$.
3. **Induction step** WTS for any set A of size k+1 and any finite set B, $|A \times B| = |A||B|$.
Induction step

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\}$

Let $B$ be any finite set. Then

$$A \times B = (C \times B) \cup \{(x_{k+1}, y) \mid y \text{ is in } B\}$$

so

$$|A \times B| = |C \times B| + |\{(x_{k+1}, y) \mid y \text{ is in } B\}|$$

$$= k |B| + |B| = (k+1) |B| = |A| |B|$$
Operations on sets

**Power set:** For a set $S$, its power set is the set of all subsets of $S$. \[ \mathcal{P}(S) = \{ A \mid A \subseteq S \} \]

If the set $S$ is finite then …

*Does the size of the power set of $S$ depend *just* on the size of $S?*
Building up the power set

<table>
<thead>
<tr>
<th>Set</th>
<th>Power set</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{ emptyset, {a} }</td>
</tr>
<tr>
<td>{a,b}</td>
<td>{ emptyset, {a}, {b}, {a,b} }</td>
</tr>
<tr>
<td>{a,b,c}</td>
<td>{ emptyset, {a}, {b}, {a,b}, {c}, {a,c}, {b,c}, {a,b,c} }</td>
</tr>
</tbody>
</table>

*Observe:* the size of the power set increases by a factor of 2 for every new element of the set.

$$|\mathcal{P}(A)| = 2^{|A|}$$
How do we prove this?

**Theorem:** For any nonnegative integer \( n \), if \( A \) is any set of size \( n \), then the power set of \( A \) has size \( 2^n \).

**Proof by Mathematical Induction:**

1. **Basis step** WTS if \( A \) is empty, then its power set has size \( 2^0 \).
2. **Inductive hypothesis** Let \( k \) be a nonnegative integer. Assume that the power set of any set \( C \) of size \( k \) has size \( 2^k \).
3. **Induction step** WTS for any set \( A \) of size \( k+1 \), its power set will have size \( 2^{k+1} \).
Induction step

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\}$

What are the subsets of $A$?
Mathematical induction* 

Rosen p. 311

To show that some statement $P(k)$ is true about all nonnegative integers $k \geq b$,

1. Show that it’s true about $b$  
   i.e. $P(b)$
2. Show $\forall k \ ( P(k) \rightarrow P(k + 1) )$  
   Hence conclude $P(b+1)$,

...
Recursive definitions

Rosen Sec 5.3

Define a set $S$ by

\[
\{ \ldots \} \\
\{ x \mid P(x) \}
\]

**Recursive definition:**

- **Basis step** – Specify initial collection of elements.
- **Recursive step** – Provide rules for forming new elements in the set from those already known to be in the set.

Induction: every nonnegative integer can be reached by starting at 0 and \textbf{adding 1} finitely many times.
Let $S$ be the subset of the set of integers defined recursively by

**Basis step:** $1 \in S$

**Recursive step:** If $a \in S$, then $a + 2 \in S$.

What's an equivalent description of this set?

A. All positive multiples of 2.
B. All positive even integers.
C. All positive odd integers.
D. All integers.
E. None of the above.
Recursive definitions

The set of **bit strings** \(\{0, 1\}^*\) is defined recursively by

*Basis step:* \(\lambda \in \{0, 1\}^*\) where \(\lambda\) is the empty string.

*Recursive step:* If \(w \in \{0, 1\}^*\), then \(w0 \in \{0, 1\}^*\) and \(w1 \in \{0, 1\}^*\).

Which of the following are **not** bit strings?

A. \(\lambda\)
B. 1
C. 000
D. 010101
E. 101100111000111100001111100000111…
Structural induction

To show that some statement $P(k)$ is true about all elements of a recursively defined set $S$,

1. Show that it's true for each element specified in the basis step to be part of $S$.
2. Show that if it's true for each of the elements used to construct new elements in recursive step, then it holds for these new elements.
Define the subset \( S \) of the set of all bit strings, \( \{0,1\}^* \), by

**Basis step:** \( \lambda \in S \) where \( \lambda \) is the empty string.

**Recursive step:** If \( w \in S \), then each of \( 10w \in S, 01w \in S \).

After the basis step and one application of the recursive step, which of these bit strings are guaranteed to be elements in \( S \)?

A. 0
B. 1
C. 01
D. 0101
E. None of the above
Define the subset $S$ of bit strings $\{0,1\}^*$ by

**Basis step:** $\lambda \in S$ where $\lambda$ is the empty string.

**Recursive step:** If $w \in S$, then each of $10w \in S$, $01w \in S$.

**Claim:** Every element in $S$ has an equal number of 0s and 1s.
Define the subset $S$ of bit strings $\{0,1\}^*$ by

**Basis step:** $\lambda \in S$ where $\lambda$ is the empty string.

**Recursive step:** If $w \in S$, then each of $10w \in S$, $01w \in S$

**Claim:** Every element in $S$ has an equal number of 0s and 1s.

**Proof:**
- **Basis step** – WTS that empty string has equal # of 0s and 1s
- **Recursive step** – Let $w$ be an arbitrary element of $S$. Assume, as the **IH** that $w$ has equal # of 0s and 1s.
  WTS that $10w$, $01w$ each have equal # of 0s, 1s.
Recursive definitions, part 2

For functions \( f: \mathbb{N} \rightarrow X \)
- define by a (closed-form) formula or …

Sequences are functions too!

To specify a function, we need to specify its

1. domain – input
2. codomain – type of outputs
3. assignment / rule – formula, table of values, induction

Rosen p. 138
Recursive definitions, part 2

For functions \( f: \mathbb{N} \rightarrow X \)
- define by a (closed-form) formula \or\ …

**Basis step:** Specify the value of the function at 0

**Recursive step:** Give a rule for finding its value at an integer from its values at smaller integers

Which of the following functions / sequences have recursive definitions?

A. \( n! \)
B. \( 2^n \)
C. \( 2, -8, 32, -128, 512, \ldots \)

D. \( \sum_{i=1}^{n} (i^2 + i) \)

E. All of the above.
Recursive definitions, part 2

\[
n! \quad 0! = 1, \quad n! = n \,(n-1)! \quad \text{for } n > 0.
\]

\[
2^n \quad 2^0 = 1, \quad 2^n = 2 \,(2^{n-1}) \quad \text{for } n > 0.
\]

\[
2, \ -8, \ 32, \ -128, \ 512, \ ...
\]

\[
a_0 = 2, \quad a_n = -4 \, a_{n-1} \quad \text{for } n > 0.
\]

\[
\sum_{i=1}^{n} g(n) \quad \sum_{i=1}^{0} g(i) = 0 \quad , \quad \sum_{i=1}^{1} g(i) = g(1) \quad , \quad \text{and} \quad \sum_{i=1}^{n} g(i) = g(n) + \sum_{i=1}^{n-1} g(i) \quad \text{for } n>1.
\]
Sigma notation for sums

You proved:

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

\[ \sum_{i=1}^{n} \frac{1}{i(i + 1)} = \frac{n}{n + 1} \]