Today's learning goals

• Explain the steps in a proof by mathematical induction
• Use mathematical induction to prove
  • correctness of identities and inequalities
  • properties of algorithms
  • properties of geometric constructions
Induction: a road map

• Today
  • What is a proof by induction?
  • Examples: inequalities, algorithms, constructions

• Tuesday
  • Strong induction
  • Recursive definitions: functions, sets, sigma notation
  • More examples / proofs: identities, constructions
Proof strategies so far

**Theorem:** \( \forall x P(x) \) over a given domain.

**Strategy (1):** Let \( x \) be arbitrary element of the domain. \textit{WTS} \( P(x) \) is true.

**Strategy (2) if domain finite:** Enumerate all \( x \) in domain. \textit{WTS} \( P(x) \) is true.

**Strategy (3) Proof by contradiction:** Assume there is an \( x \) with \( P(x) \) false. \textit{WTS} badness!

**Theorem:** \( \exists x P(x) \) over a given domain.

**Strategy (1):** Define \( x = \ldots \) (some specific element in domain) \textit{WTS} \( P(x) \) is true.

**Strategy (2) Proof by contradiction:** Assume that for all \( x \), \( P(x) \) is false. \textit{WTS} badness!

**Theorem:** \( P \rightarrow Q \) over a given domain.

**Strategy (1):** Toward direct proof, assume \( P \) and \textit{WTS} \( Q \).

**Strategy (2):** Toward proof by contrapositive, assume \( Q \) is false and \textit{WTS} \( P \) is also false.

**Strategy (3) Proof by contradiction:** Assume both \( P \) is true and \( Q \) is false. \textit{WTS} badness!
A new proof strategy

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it’s true about 0 i.e. $P(0)$
2. Show $P(0) \rightarrow P(1)$ Hence conclude $P(1)$
3. Show $P(1) \rightarrow P(2)$ Hence conclude $P(2)$
4. Show $P(2) \rightarrow P(3)$ Hence conclude $P(3)$
5. .....
A new proof strategy

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it's true about 0, i.e. $P(0)$
2. Show $P(0) \Rightarrow P(1)$
3. Show $P(1) \Rightarrow P(2)$
4. Show $P(2) \Rightarrow P(3)$
5. ....

Need both for induction to be applicable
Mathematical induction

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it’s true about 0 i.e. $P(0)$
2. Show $\forall k \ P(k) \rightarrow P(k + 1)$ Hence conclude $P(1), \ldots$
An inequality \[ \left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}\]

Theorem: This inequality is true for all nonnegative integers

Proof: by Mathematical Induction.

What's P(n)?

A. \[\left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2}\]

B. \[\left(1 + \frac{1}{2}\right)^k \geq 1 + \frac{n}{2}\]

C. \[\left(1 + \frac{1}{2}\right)^k\]

D. \[\left(1 + \frac{n}{2}\right)^k\]

E. None of the above.
An inequality \((1 + \frac{1}{2})^n \geq 1 + \frac{n}{2}\)

**Theorem:** This inequality is true for all nonnegative integers

**Proof:** by Mathematical Induction.

1. **Basis step** WTS \((1 + \frac{1}{2})^0 \geq 1 + \frac{0}{2}\)

2. **Inductive hypothesis** Let \(k\) be a nonnegative integer. Assume \((1 + \frac{1}{2})^k \geq 1 + \frac{k}{2}\)

3. **Induction step** WTS \((1 + \frac{1}{2})^{k+1} \geq 1 + \frac{k+1}{2}\)
Robot

Start at origin, moves on infinite 2-dimensional integer grid. At each step, move to diagonally adjacent grid point.

Can it ever reach (1,0)?
Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof of Lemma: by mathematical induction on the number of steps.

Using proof: Sum of coordinates of (1,0) is 1 so not even!
Proof of Invariant

Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof: by Mathematical Induction.

1. **Basis step** WTS The sum of the coordinates reachable after 0 steps is even.

2. **Inductive hypothesis** Let $k$ be a nonnegative integer. Assume the sum of the coordinates of any state reachable after $k$ steps is even.

3. **Induction step** WTS the sum of the coordinates of any state reachable after $k+1$ steps is even.
Theorem: If start with an equilateral triangle and divide each side into n equal segments, then connect the division points with all possible line segments parallel to sides of original triangle, then ______ many small triangles will be contained in the original triangle.
Theorem: At stage $n$, $n^2$ triangles are formed.

Proof: by Mathematical Induction.

1. **Basis step** WTS that when side lengths undivided, $1^2$ triangles are formed.

2. **Inductive hypothesis** Let $k$ be a nonnegative integer. Assume that when divide sides into $k$ pieces, $k^2$ triangles are formed.

3. **Induction step** WTS when divide sides into $k+1$ pieces, $(k+1)^2$ triangles are formed.
Sizes of (finite) sets

If S is a set with exactly \( n \) distinct elements, with \( n \) a nonnegative integer, then S is finite set and \(|S| = n\).

Which of the following sets are finite?

Assume universe is set of real numbers.

A. \( \emptyset \)
B. \( \mathbb{Q} \cap \{x \mid 0 \leq x \leq 1\} \)
C. \( \mathbb{Z} \cap \{x \mid 0 \leq x \leq 1\} \)
D. \( \mathbb{Z} \cup \{x \mid 0 \leq x \leq 1\} \)
E. None of the above.

An infinite set is a set that is not finite.
Operations on sets

If the sets $A$, $B$ are finite then

$$|A \cup B| = ?$$

A. $|A| + |B|$
B. $|A| - |B|$
C. $|A| \cdot |B|$
D. $|A|^{|B|}$
E. None of the above.

Rosen Sections 2.1, 2.2
Operations on sets

If the sets A, B are finite then

\[|A \cup B| = |A| + |B| - |A \cap B|\]
Operations on sets

If the sets $A$, $B$ are finite then

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

What's the size of the difference $A-B$?
A. $|A \cup B| - |A \cap B|$
B. $|A \cup B| - |A| - |B|$
C. $|A| - |A \cap B|$
D. $|A| - |B|$
E. None of the above.
Two sets $A$ and $B$ are **disjoint** iff

$$A \cap B = \emptyset$$

Which of the following is *not* an equivalent characterization of $A$ and $B$ being disjoint?

A. $|A \cup B| = |A| + |B|$
B. $A - B = A$
C. $A \subseteq B$
D. $\exists x(x \in A \land x \in B)$
E. None of the above.
Operations on sets

If the sets $A$, $B$ are finite then

$$|A \times B| = |A| \cdot |B|$$

$|B|$ many elements for each of the $|A|$ many elements in $A$
How do we prove this general formula?

\[ |A \times B| = |A| \cdot |B| \]

How much does \(|A \times B|\) change when we add one new element to \(A\)?

A. Add one element.
B. Add \(|B|\) many elements.
C. Multiply number of elements by \(|B|\).
D. Raise number of elements to \(|B|^\text{th} \) power.
E. None of the above.
How do we prove this?

**Theorem:** For any nonnegative integer $n$, if $A$ is any set of size $n$, then for any finite set $B$, $|A \times B| = |A||B|$.

**Proof by Mathematical Induction:**

1. **Basis step** WTS if $A$ is empty, then for any finite set $B$, $|A \times B| = |A||B| = 0$.

2. **Inductive hypothesis** Let $k$ be a nonnegative integer. Assume that for any set $C$ of size $k$ and any finite set $D$, $|C \times D| = |C||D|$.

3. **Induction step** WTS for any set $A$ of size $k+1$ and any finite set $B$, $|A \times B| = |A||B|$.
Induction step

Write \( A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\} \)

Let \( B \) be any finite set. Then

\[
A \times B = (C \times B) \cup \{(x_{k+1}, y) \mid y \text{ is in } B\}
\]

so

\[
|A \times B| = |C \times B| + |B| = k |B| + |B| = (k+1) |B| = |A| |B|
\]
Operations on sets

Rosen Sections 2.1, 2.2

Power set: For a set $S$, its power set is the set of all subsets of $S$.

$\mathcal{P}(S) = \{ A \mid A \subseteq S \}$

If the set $S$ is finite then …

Does the size of the power set of $S$ depend just on the size of $S$?
# Building up the power set

<table>
<thead>
<tr>
<th>Set</th>
<th>Power set</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{ emptyset, {a} }</td>
</tr>
<tr>
<td>{a, b}</td>
<td>{ emptyset, {a}, {b}, {a, b} }</td>
</tr>
<tr>
<td>{a, b, c}</td>
<td>{ emptyset, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c} }</td>
</tr>
</tbody>
</table>

*Observe*: the size of the power set increases by a factor of 2 for every new element of the set.

\[ |\mathcal{P}(A)| = 2^{|A|} \]
How do we prove this?

**Theorem**: For any nonnegative integer $n$, if $A$ is any set of size $n$, then the power set of $A$ has size $2^n$.

**Proof by Mathematical Induction:**

1. **Basis step** WTS if $A$ is empty, then its power set has size $2^0$.
2. **Inductive hypothesis** Let $k$ be a nonnegative integer. Assume that the power set of any set $C$ of size $k$ has size $2^k$.
3. **Induction step** WTS for any set $A$ of size $k+1$, its power set will have size $2^{k+1}$. 
Induction step

Write \( A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\} \)

What are the subsets of \( A \)?