Problem 1

Recall that the projection of a vector $x$ onto another vector $y$ is defined as the vector \( \frac{\langle x, y \rangle}{\|y\|^2} y \). Suppose $u_1 \neq u_2$ are two vectors such that $\|u_1\| = \|u_2\| = 1$, and $u_1$ and $u_2$ are not orthogonal to each other. Let $x$ be a third non-zero vector.

Alice and Bob have been asked to take projections of $x$ onto $u_1$ and $u_2$. Alice first projects $x$ onto $u_1$ and then projects the result onto $u_2$ to get the vector $a$. Bob first projects $x$ onto $u_2$ and then projects the result onto $u_1$ to get the vector $b$.

1. Is $\|a\| = \|b\|$ for all $x$? Justify your answer if this is the case, and provide a counter-example if this is not the case.

2. Now suppose that $u_1$ and $u_2$ are orthogonal to each other. Does this change your answer to part (1)? Justify your answer (or provide a counterexample, as the case may be.)