Counting Strategies: Inclusion-Exclusion, Categories

<table>
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<tr>
<th>Lecture</th>
<th>Instructor</th>
<th>Time</th>
<th>Room</th>
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<tbody>
<tr>
<td>Lecture A</td>
<td>Tiefenbruck</td>
<td>MWF 9-9:50am</td>
<td>Center 212</td>
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<td>Lecture B</td>
<td>Jones</td>
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<td>Lecture C</td>
<td>Tiefenbruck</td>
<td>MWF 11-11:50am</td>
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http://cseweb.ucsd.edu/classes/wi16/cse21-abc/

February 12, 2016
Select which method lets us count the number of length n binary strings.

A. The product rule.
B. The sum rule.
C. Either rule works.
D. Neither rule works.
Select which method lets us count the number of length n binary strings.

A. The product rule. Select first bit, then second, then third …
B. The sum rule. \{0\ldots\} \cup \{1\ldots\} gives recurrence \( N(n) = 2N(n-1), \ N(0)=1 \)
C. Either rule works.
D. Neither rule works.
Memory: storing length $n$ binary strings

How many binary strings of length $n$ are there?

How many bits does it take to store a length $n$ binary string?
Memory: storing length $n$ binary strings

How many binary strings of length $n$ are there? $2^n$

How many bits does it take to store a length $n$ binary string? $n$

**General principle:** number of bits to store an object is

$$\lceil \log_2(\text{number of objects}) \rceil$$

Why the ceiling function?
Memory: storing integers

Scenario: We want to store a non-negative integer that has at most $n$ digits. How many bits of memory do we need to allocate?

A. $n$
B. $2^n$
C. $10^n$
D. $n \times \log_2 10$
E. $n \times \log_{10} 2$
Ice cream!

At an ice cream parlor, you can choose to have your ice cream in a bowl, cake cone, or sugar cone. There are 20 different flavors available.

How many single-scoop creations are possible?

A. 20  
B. 23  
C. 60  
D. 120  
E. None of the above.
Ice cream!

At an ice cream parlor, you can choose to have your ice cream in a bowl, cake cone, or sugar cone. There are 20 different flavors available.

You can convert your single-scoop of ice cream to a sundae. Sundaes come with your choice of caramel or hot-fudge. Whipped cream and a cherry are options. How many desserts are possible?

A. 20*3*2*2
B. 20*3*2*2*2
C. 20*3 + 20*3*2*2
D. 20*3 + 20*3*2*2*2
E. None of the above.
A scheduling problem

In one request, four jobs arrive to a server: J1, J2, J3, J4.

The server starts each job right away, splitting resources among all active ones.

Different jobs take different amounts of time to finish.

How many possible finishing orders are there?

A. $4^4$
B. $4+4+4+4$
C. $4 \times 4$
D. None of the above.
A scheduling problem

In one request, four jobs arrive to a server: J1, J2, J3, J4.

The server starts each job right away, splitting resources among all active ones.

Different jobs take different amounts of time to finish.

**How many possible finishing orders are there?**

*Product rule analysis*

- 4 options for which job finishes first.
- Once pick that job, 3 options for which job finishes second.
- Once pick those two, 2 options for which job finishes third.
- Once pick first three jobs, only 1 remains.

\[(4)(3)(2)(1) = 4! = 24\]
Permutations

**Permutation:** rearrangement / ordering of n distinct objects so that each object appears exactly once

\[1, 2, 3, J, 4\]

**Theorem 1:** The number of permutations of n objects is

\[n! = n(n-1)(n-2) \ldots (3)(2)(1)\]

**Convention:** \(0! = 1\)
Traveling salesperson

Planning a trip to

New York
Chicago
Baltimore
Los Angeles
San Diego
Minneapolis
Seattle

Must start in New York and end in Seattle.

How many ways can the trip be arranged?

A. $7!$
B. $2^7$
C. None of the above.
Traveling salesperson

Planning a trip to

New York
Chicago
Baltimore
Los Angeles
San Diego
Minneapolis
Seattle

Must **start in New York** and **end in Seattle**.
Must also **visit Los Angeles immediately after San Diego**.

*How many ways can the trip be arranged now?*
Traveling salesperson

Planning a trip to

New York
Chicago
Baltimore
Los Angeles
San Diego
Minneapolis
Seattle

Must start in New York and end in Seattle.
Must also visit Los Angeles immediately after San Diego.

How many ways can the trip be arranged now?

Treat LA & SD as a single stop.

$(1)(4!)(1) = 24$ arrangements.
Traveling salesperson

Planning a trip to

New York
Chicago
Baltimore
Los Angeles
San Diego
Minneapolis
Seattle

Must start in New York and end in Seattle.
Must also visit Los Angeles and San Diego immediately after each other (in any order).

How many ways can the trip be arranged now?
Traveling salesperson

Planning a trip to

New York
Chicago
Baltimore
Los Angeles
San Diego
Minneapolis
Seattle

Must start in New York and end in Seattle.
Must also visit Los Angeles and San Diego immediately after each other (in any order).

How many ways can the trip be arranged now?

Break into two disjoint cases:
Case 1: LA before SD 24 arrangements
Case 2: SD before LA 24 arrangements
Traveling salesperson

Planning a trip to

New York
Chicago
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Seattle

Must start in New York and end in Seattle.
Must also visit Los Angeles and San Diego immediately after each other (in any order).

How many ways can the trip be arranged now?
Traveling salesperson

Planning a trip to

New York
Chicago
Baltimore
Los Angeles
San Diego
Minneapolis
Seattle

Is there an order of visiting the cities that stops at each city exactly once and minimizes the distance traveled?

<table>
<thead>
<tr>
<th></th>
<th>NY</th>
<th>Chicago</th>
<th>Balt.</th>
<th>LA</th>
<th>SD</th>
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<td>1700</td>
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</tbody>
</table>
Traveling salesperson

Planning a trip to

New York
Chicago
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Los Angeles
San Diego
Minneapolis
Seattle

Is there an order of visiting the cities that stops at each city exactly once and minimizes the distance traveled?

Want a Hamiltonian tour
Traveling salesperson

Developing an algorithm which, given a set of cities and distances between them, computes a shortest distance path between all of them is **NP-hard** (considered intractable, very hard).

Is there **any** algorithm for this question?

A. No, it's not possible.
B. Yes, it's just very slow.
C. ?

Want a Hamiltonian tour
Traveling salesperson

Exhaustive search algorithm

List all possible orderings of the cities. For each ordering, compute the distance traveled. Choose the ordering with minimum distance.

How long does this take?

Want a Hamiltonian tour
Traveling salesperson

Exhaustive search algorithm: given $n$ cities and distances between them.

List all possible orderings of the cities. For each ordering, compute the distance traveled. Choose the ordering with minimum distance.

How long does this take?

Want a Hamiltonian tour
Traveling salesperson

Exhaustive search algorithm: given \( n \) cities and distances between them.

List all possible orderings of the cities.
For each ordering, compute the distance traveled. 
Choose the ordering with minimum distance.

\[ O(\text{number of orderings}) \]

How long does this take?

A. \( O(n) \)
B. \( O(n^2) \)
C. \( O(n^n) \)
D. \( O(n!) \)
E. None of the above.
Traveling salesman

Exhaustive search algorithm: given \( n \) cities and distances between them.

List all possible orderings of the cities.
For each ordering, compute the distance traveled.
Choose the ordering with minimum distance.

\[
\text{Number of orderings} = n! \
\]

How long does this take?

A. \( \mathcal{O}(n) \)
B. \( \mathcal{O}(n^2) \)
C. \( \mathcal{O}(n^n) \)
D. \( \mathcal{O}(n!) \)
E. None of the above.

\[
2^n < n! < n^n \quad \text{for large } n
\]

Moral: counting gives upper bound on algorithm runtime.
A complete bipartite graph is an undirected graph whose vertex set is partitioned into two sets $V_1, V_2$ such that

- there is an edge between each vertex in $V_1$ and each vertex in $V_2$
- there are no edges both of whose endpoints are in $V_1$
- there are no edges both of whose endpoints are in $V_2$

A complete bipartite graph is an undirected graph whose vertex set is partitioned into two sets $V_1, V_2$ such that

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- there are no edges both of whose endpoints are in $V_2$

**Is this graph Hamiltonian?**

A. Yes
B. No
A **complete bipartite graph** is an undirected graph whose vertex set is partitioned into two sets $V_1, V_2$ such that

- there is an edge between each vertex in $V_1$ and each vertex in $V_2$
- there are no edges both of whose endpoints are in $V_1$
- there are no edges both of whose endpoints are in $V_2$

Rosen p. 658

**Is every complete bipartite graph Hamiltonian?**

A. Yes
B. No
Bipartite Graphs

Claim: any complete bipartite graph with $|V_1| = k$, $|V_2| = k+1$ is Hamiltonian.

How many Hamiltonian tours can we find?

A. $k$
B. $k(k+1)$
C. $k!(k+1)!$
D. $(k+1)!$
E. None of the above.
**Claim:** any complete bipartite graph with $|V_1| = k$, $|V_2| = k+1$ is Hamiltonian.

**How many Hamiltonian tours can we find?**

A. $k$
B. $k(k+1)$
C. $k!(k+1)!$
D. $(k+1)!$
E. None of the above.

*Product rule!*
When product rule fails

How many Hamiltonian tours can we find?

A. 5!
B. 5!4!
C. ?

5

depends on
previous choices

first

next

---

How many Hamiltonian tours can we find?

A. 5!
B. 5!4!
C. ?

all reorderings of a, b, c, d, e

\( e \cdot b \cdot c \cdot d \cdot a \)
When product rule fails

Tree Diagrams

Which Hamiltonian tours start at e?
List all possible next moves.
Then count leaves.

Rosen p.394-395
When sum rule fails

Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

How many people know Java or C (or both)?

A. $|A| + |B|$
B. $|A| \cdot |B|$
C. $|A| \cdot |B|$
D. $|B| \cdot |A|$
E. None of the above.

Rosen p. 392-394
When sum rule fails

Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

# people who know Java or C = # people who know Java
When sum rule fails

Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

# people who know Java or C = # people who know Java + # people who know C

Double counted!
When sum rule fails

Let A = { people who know Java } and B = { people who know C }

# people who know Java or C = # people who know Java 
+ # people who know C 
- # people who know both

Rosen p. 392-394
Inclusion-Exclusion principle

Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Include $A$, Include $B$, Exclude $A \cap B$
Inclusion-Exclusion for three sets

\[ A \cup B \cup C \] = ?

Rosen p. 392-394
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]

Rosen p. 392-394
Inclusion-Exclusion for three sets

$|A \cup B \cup C| = ?$

Rosen p. 392-394
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]

Rosen p. 392-394
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]

Rosen p. 392-394
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| =? \]
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]

Rosen p. 392-394
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C| \]

Rosen p. 392-394
Inclusion-Exclusion principle

If $A_1, A_2, \ldots, A_n$ are finite sets then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k|$$

$\ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$
How many four-letter strings have one vowel and three consonants?

There are 5 vowels: AEIOU and 21 consonants: BCDFGHJKLMNPQRSTVWXYZ.

A. $5 \times 21^3$  
B. $26^4$  
C. $5 + 52$  
D. None of the above.

\[
\frac{5 \times 21 \times 21 \times 21}{5} = \frac{21 \times 21 \times 21}{c} \times \frac{c}{c} \times \frac{c}{c} \times \frac{c}{c} \text{ (product rule)}
\]
How many four-letter strings have one vowel and three consonants?

There are 5 vowels: AEIOU and 21 consonants: BCDFGHJKLMNPQRSTVWXYZ.

<table>
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<tr>
<th>Template</th>
<th># Matching</th>
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<tr>
<td>VCCC</td>
<td>$5 \times 21 \times 21 \times 21$</td>
</tr>
<tr>
<td>CVCC</td>
<td>$21 \times 5 \times 21 \times 21$</td>
</tr>
<tr>
<td>CCVC</td>
<td>$21 \times 21 \times 5 \times 21$</td>
</tr>
<tr>
<td>CCCV</td>
<td>$21 \times 21 \times 21 \times 5$</td>
</tr>
</tbody>
</table>

Total: $4 \times 5 \times 21^3$

In total, use the sum rule.
If \( A = X_1 \cup X_2 \cup \ldots \cup X_n \) and all \( X_i, X_j \) disjoint and all \( X_i \) have same size, then

\[ |X_i| = \frac{|A|}{n} \]

More generally:

There are \( \frac{n}{d} \) ways to do a task if it can be done using a procedure that can be carried out in \( n \) ways, and for every way \( w \), \( d \) of the \( n \) ways give the same result as \( w \) did.
If $A = X_1 \cup X_2 \cup \ldots \cup X_n$ and all $X_i$, $X_j$ disjoint and all $X_i$ have same size, then

$$|X_i| = |A| / n$$

More generally:

There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, $d$ of the $n$ ways give the same result as $w$ did.
If $A = X_1 \cup X_2 \cup \ldots \cup X_n$ and all $X_i, X_j$ disjoint and all $X_i$ have same size, then

$$|X_i| = \frac{|A|}{n}$$

Or in other words,

If objects are partitioned into categories of equal size, and we want to think of different objects as being the same if they are in the same category, then

$$n = \frac{|A|}{\left| X_i \right|}$$

# categories = (# objects) / (size of each category)
An ice cream parlor has n different flavors available. How many ways are there to order a two-scoop ice cream cone (where you specify which scoop goes on bottom and which on top, and the two flavors must be different)?

A. \( n^2 \)
B. \( n! \)
C. \( n(n-1) \)
D. \( 2n \)
E. None of the above.
An ice cream parlor has \( n \) different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

A. Double the previous answer.
B. Divide the previous answer by 2.
C. Square the previous answer.
D. Keep the previous answer.
E. None of the above.
Ice cream!

An ice cream parlor has n different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

Objects:
Categories:
Size of each category:

\[ \text{# categories} = \frac{\text{(# objects)}}{\text{(size of each category)}} \]
An ice cream parlor has \( n \) different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:** cones  
**Categories:** flavor pairs (regardless of order)  
**Size of each category:**

\[
\# \text{ categories} = \frac{\# \text{ objects}}{\text{size of each category}}
\]
An ice cream parlor has $n$ different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:** cones $n(n-1)$

**Categories:** flavor pairs (regardless of order)

**Size of each category:** 2

$$\text{# categories} = \frac{n(n-1)}{2}$$

Avoiding double-counting
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

A. 3!
B. $2^3$
C. $3^2$
D. 1
E. None of the above.
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

**Objects**: all different colored triangles  
**Categories**: physical colored triangles (two triangles are the same if they can be rotated and/or flipped to look alike)  
**Size of each category**:  

\[
\text{# categories} = \left(\text{# objects}\right) / (\text{size of each category})
\]
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

**Objects**: all different colored triangles \(3!\)

**Categories**: physical colored triangles (two triangles are the same if they can be rotated and/or flipped to look alike)

**Size of each category**: \((3)(2)\) three possible rotations, two possible flips

\[
\# \text{ categories} = \frac{\# \text{ objects}}{(\text{size of each category})} = \frac{6}{6} = 1
\]
Reminders

HW 6 due **Wednesday 11:59pm** via **Gradescope**.

Monday is a holiday. No lecture, discussion, or office hours.