CSE 20
DISCRETE MATH

WINTER 2016

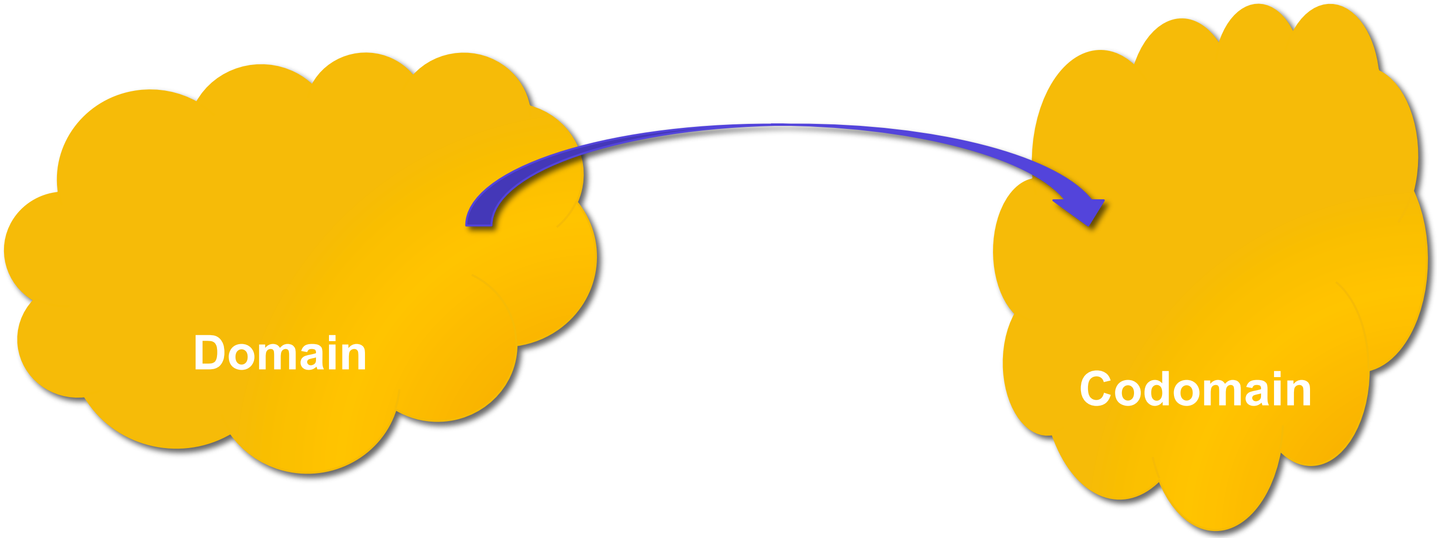
http://cseweb.ucsd.edu/classes/wi16/cse20-ab/

Today's learning goals

- Represent functions in multiple ways
- Define and prove properties of: domain of a function, image of a function, composition of functions
- Determine and prove whether a function is one-to-one, onto, bijective
- Apply the definition and properties of floor function, ceiling function, factorial function
- Define and compute the cardinality of a set: Finite sets, countable sets, uncountable sets
- Use functions to compare the sizes of sets
- Use functions to define sequences: arithmetic progressions, geometric progressions
- Use and prove properties of recursively defined functions and recurrence relations (using induction)
- Use and interpret Sigma notation

Functions

Rosen p. 138



Function **Mapping** **Transformation**

$$\forall a (a \in D \to \exists! b \, (b \in C \land f(a) = b))$$

To specify a function

(1) Domain (2) Codomain (3) Assignment

Operations on functions

Rosen p. 141,147

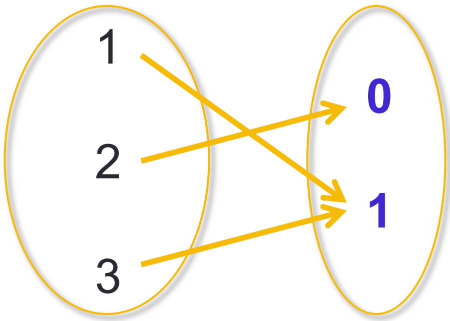
If $f: A \rightarrow \mathbf{R}$, $g: A \rightarrow \mathbf{R}$ $f+g: A \rightarrow \mathbf{R}$ $fg: A \rightarrow \mathbf{R}$

If $f: B \rightarrow C$, $g: A \rightarrow B$ $f \circ g: A \rightarrow C$

Properties of functions

possible

- A function f is ……….. means **at least one input for every output**



How can we formalize this?

A. $\forall a(a \in D \rightarrow \exists! b\,(b \in C \wedge f(a) = b))$

B. $\forall b(b \in C \rightarrow \exists! a\,(a \in D \wedge f(a) = b))$

C. $\forall a \forall b((a \in D \wedge b \in C) \rightarrow f(a) = b)$

D. $\forall b(b \in C \rightarrow \exists a\,(a \in D \wedge f(a) = b))$

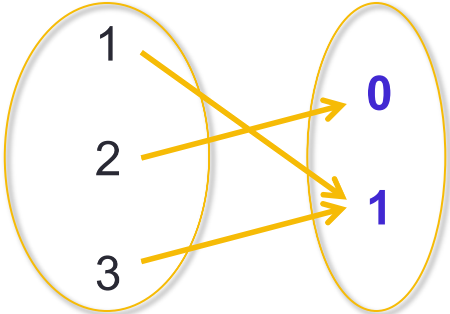
E. None of the above

Properties of functions

Rosen p. 143

possible

- A function f is **onto** (surjective) means **at least one input for every output**

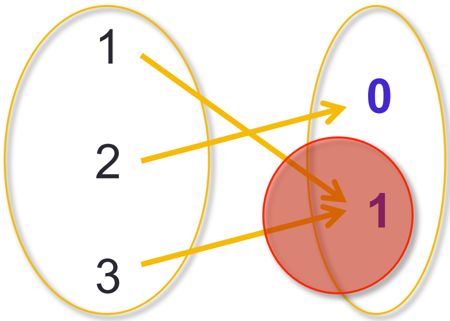


$$\forall b(b \in C \to \exists a\,(a \in D \land f(a) = b))$$

Properties of functions

Rosen p. 141

- A function f is **one-to-one** means **no duplicate images** (injective)



How can we formalize this?

A. $\forall a \forall b((a \in D \wedge b \in D) \rightarrow f(a) \neq f(b))$

B. $\forall a \forall b((a \in D \wedge b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$

C. $\forall a \forall b((a \in C \wedge b \in C) \rightarrow a \neq b)$

D. $\forall a \forall b((a \in C \wedge b \in C) \rightarrow f(a) \neq f(b))$

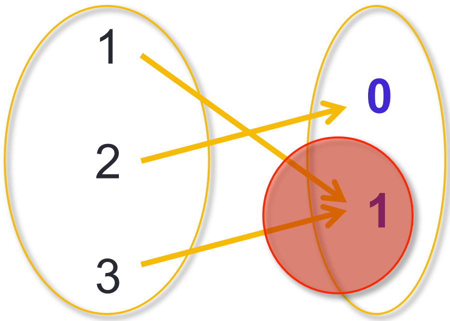
E. None of the above

Properties of functions

Rosen p. 141

- A function f is **one-to-one** means **no duplicate images**

  (injective)

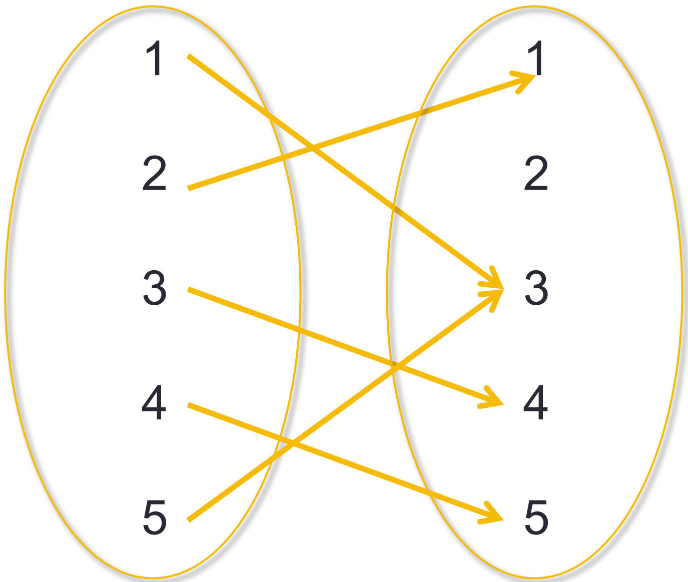


$$\forall a \forall b((a \in D \land b \in D) \rightarrow (f(a) = f(b) \rightarrow a = b))$$

$$\forall a \forall b((a \in D \land b \in D) \rightarrow (a \neq b \rightarrow f(a) \neq f(b)))$$

Onto? One-to-one?

Consider the function over domain and codomain {1,2,3,4,5} defined by



This function is

A. Well defined, onto, and one-to-one.
B. Well defined, but neither onto nor one-to-one.
C. Well defined, onto, but not one-to-one.
D. Not well-defined, not onto, not one-to-one.
E. None of the above.

Onto? One-to-one?

Consider the function over domain and codomain $\mathbf{R}^{>=0}$ defined by

$f(x) = x^2$

This function is

A. Well defined, onto, and one-to-one.
B. Well defined, but neither onto nor one-to-one.
C. Well defined, onto, but not one-to-one.
D. Not well-defined, not onto, not one-to-one.
E. None of the above.

Proving a function is …

Define f:{0,1}* → **N** by f(w) = |w|, or formally, f is defined recursively by

$$\begin{cases} f(\lambda) = 0 \\ f(w0) = f(w) + 1 \\ f(w1) = f(w) + 1 \end{cases}$$

Fact: This function is onto.

Proving a function is …

Define f:{0,1}* → **N** by f(w) = |w|, or formally, f is defined recursively by

$$\begin{cases} f(\lambda) = 0 \\ f(w0) = f(w) + 1 \\ f(w1) = f(w) + 1 \end{cases}$$

Fact: This function is not one-to-one.

Proving a function is …

Let A = {1,2,3} and B = {2,4,6}.

Define a function from the power set of A to the power set of B by:

$$f : \mathcal{P}(A) \to \mathcal{P}(B)$$
$$f(X) = X \cap B$$

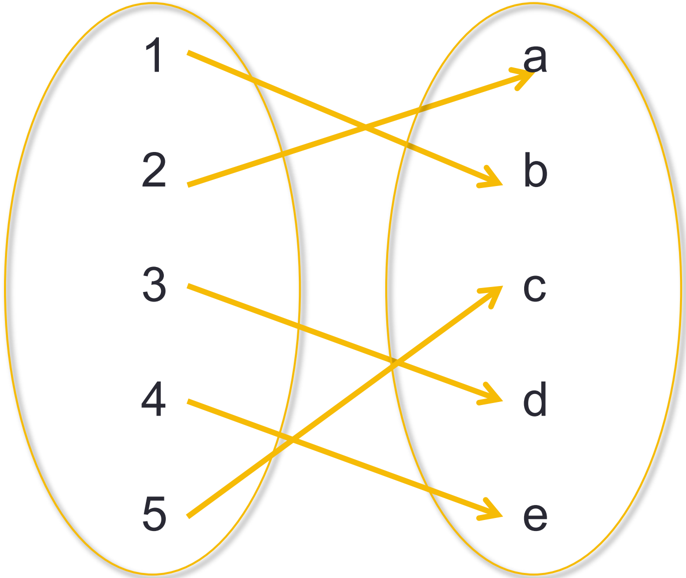
Well-defined?

Onto?

One-to-one?

One-to-one + onto

Rosen p. 144



one-to-one correspondence

bijection

invertible

The **inverse** of a function f: A→B is the function g: B→A such that

$$\forall b(b \in B \rightarrow (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144

What's the inverse function of

$$f(x) = x^3 + 1 \text{ ?}$$

A. $g(x) = x^{1/3} - 1$

B. $g(x) = (x - 1)^{1/3}$

C. $g(x) = \dfrac{1}{x^3 + 1}$

D. More than one of the above.

E. None of the above.

one-to-one correspondence

bijection

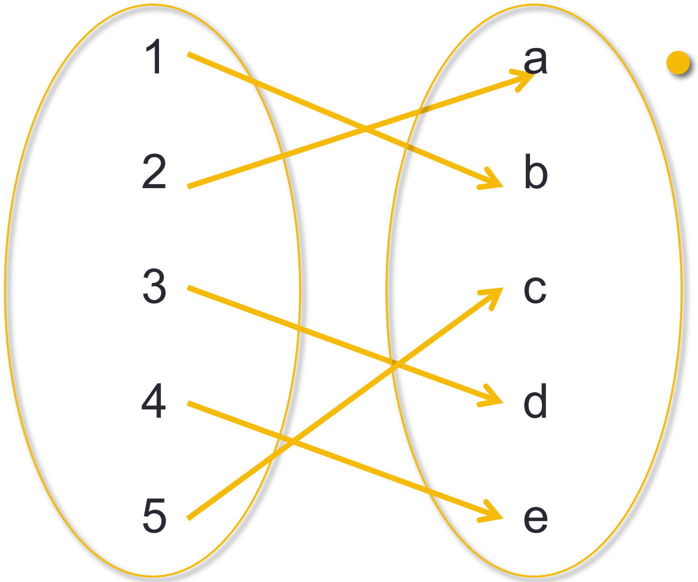
invertible

The **inverse** of a function f: A→B is the function g: B→A such that

$$\forall b(b \in B \to (g(b) = a \leftrightarrow f(a) = b))$$

One-to-one + onto

Rosen p. 144



Fact: for finite sets A and B, there is a bijection between them if and only if |A| =|B|.

Beyond finite sets

Rosen Section 2.5

For all sets, we say

|A| = |B| if and only if there is a bijection between them.

A. |**Z**| = |**N**|
B. |**N**| = |**Q**|
C. |**Z**| = |{0,1}*|
D. All of the above.
E. None of the above.

Beyond finite sets

For all sets, we say

$|A| = |B|$ if and only if there is a bijection between them.

0, 1, 2, 3, 4, 5, 6, ….

0, -1, 1, -2, 2, -3, 3, …

0, 1, -1, ½, - ½, …

λ, 0, 1, 00, 01, 10, 11, …

Beyond finite sets

For all sets, we say

|A| = |B| if and only if there is a bijection between them.

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0, -1, 1, -2, 2, -3, 3, …

0, 1, -1, ½, - ½, …

λ, 0, 1, 00, 01, 10, 11, …

Which of the following is **not** true?

A. There are sets A, B with $A \subsetneq B$ and |A| = |B|
B. There are **disjoint** sets A, B with |A| = |B|
C. There are one-to-one function that are not onto.
D. There are onto functions that are not one-to-one.
E. All infinite sets are of the same size. ←

Cardinality

Rosen Defn 3 p. 171

- Finite sets — $|A| = n$ for some nonnegative int n
- Countably infinite sets — $|A| = |\mathbf{Z}^+|$ (informally, can be listed out)
- Uncountable sets — Infinite but not in bijection with **N**