Instructions

Homework should be done in groups of one to three people. You are free to change group members at any time throughout the quarter. Problems should be solved together, not divided up between partners. A single representative of your group should submit your work through Gradescope. Submissions must be received by 11:59pm on the due date, and there are no exceptions to this rule.

Homework solutions should be neatly written or typed and turned in through Gradescope by 11:59pm on the due date. No late homeworks will be accepted for any reason. You will be able to look at your scanned work before submitting it. Please ensure that your submission is legible (neatly written and not too faint) or your homework may not be graded.

You may consult your textbook, class notes, lecture slides, instructors, TAs, and tutors for help with homework. You should not look for answers to homework problems in other texts or sources, including the internet. Only post about graded homework questions on Piazza if you suspect a typo in the assignment, or if you don’t understand what the question is asking you to do. Other questions are best addressed in office hours.

Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

For questions that require pseudocode, you can follow the same format as the textbook, or you can write pseudocode in your own style, as long as you specify what your notation means. For example, are you using “=” to mean assignment or to check equality? You are welcome to use any algorithm from class as a subroutine in your pseudocode.

Reading Rosen Sections 5.1, 5.2, 5.3, 2.3, 2.5

Key Concepts Cardinality, mathematical induction, recursive definitions, strong induction, functions, one-to-one, onto, bijection.

1. (10 points)

(a) Give a recursive definition of the function ones(s), which counts the number of ones in a bit string \( s \in \{0,1\}^* \).

Hint: the domain of this function is \( \{0,1\}^* \) and its codomain is the set of nonnegative integers \( \mathbb{N} \).

(b) Use structural induction to prove that \( \text{ones}(st) = \text{ones}(s) + \text{ones}(t) \). Note that \( st \) refers to the concatenation of the strings \( s \) and \( t \); on the RHS the + is addition of integers.

Hint: you may find the recursive definition of string concatenation, Definition 2 on page 350, useful.

(Rosen p. 359 # 132)
2. (10 points) Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.
(Rosen p. 342 # 8)

3. (10 points) We consider several functions each of whose codomain is the set of nonnegative integers, \( \mathbb{N} \). For each of the following functions, (i) find its domain, then (ii) decide whether the function is one-to-one (and prove your answer), and (iii) decide whether the function is onto (and prove your answer).

   (a) The function that assigns to each pair of positive integers the maximum of these two integers.

   (b) The function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as digits in the decimal representation of the integer.

   (c) The function that assigns to a binary string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s.

4. (10 points) Consider the set \( A = \{1, 2, 3\} \). Find a bijection between the set \( X \) of subsets of \( A \) with an odd number of elements and the set \( Y \) of subsets of \( A \) with an even number of elements: precisely define the function and prove that it is a bijection. Note: it is possible to generalize this observation to prove that every nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

5. (10 points) Give an example of two uncountable sets \( A \) and \( B \) such that \( A - B \) is

   (a) finite.

   (b) countably infinite.

   (c) uncountable.

Justify all your answers.
(Rosen p. 176 # 10)