Assignment 6 Solutions

1

(a) How many functions are there from \{1, \ldots, n\} to \{1, \ldots, k\}?

(b) A function is said to be increasing if \(a > b\) implies \(f(a) > f(b)\). How many increasing functions are there from \{1, \ldots, n\} to \{1, \ldots, k\}?

(b) A function is said to be non-decreasing if \(a > b\) implies \(f(a) \geq f(b)\). How many non-decreasing functions are there from \{1, \ldots, n\} to \{1, \ldots, k\}?

Part (a):
There are \(k^n\) functions because each element in domain has \(k\) choices among range values and there are \(n\) elements in domain.

Part(b):
Notice the following properties:

1. No two elements from domain can be mapped to same element in range.

2. If \(k < n\), we cannot have a valid function. So, \(k \geq n\).

3. If \(k = n\), then there is exactly one function that maps domain to range.

From property 1 mentioned above, there are exactly \(n\) elements in range that are mapped by any increasing function and remaining \(k - n\) elements are left out. And, from property 3, once we determine these \(n\) elements from range, there is exactly one function that maps domain to these \(n\) elements. Hence, the number of functions is same as number of ways of choosing \(n\) elements out of \(k\) from range. This is \(\binom{k}{n}\).
Part (c):
This is same as distributing \( n \) indistinguishable items to \( k \) people, that is,\( \binom{n+k-1}{n} \). The \( n \) elements in domain, arranged in increasing order, can be imagined as balls and the \( k \) elements in range can be imagined as walls. This makes the two problems identical.

2 How many 8 digit positive integers are there such that the first digit is not 0 and the integer is even and one of the 8 digits is 6.

Let us list the conditions and find corresponding constraints on counting:

1. **First digit is not 0:** This means there are only 9 choices for first digit.

2. **The integer is even:** So, the last digit should be one of 0, 2, 4, 6, 8. Hence, there are 5 choices for last digit.

3. **At least one of the digits is 6:** We shall handle this using complements.

   Complement of condition 3 is "none of the digits are 6". Hence, for our solution, we shall compute the cases satisfying only condition 1 and 2 and then subtract the cases satisfying condition 1, 2 and complement of 3.

**Number of numbers satisfying condition 1 and 2**
First digit has 9 options and last digit has 5 options. Remaining 6 digits have 10 options each. So, answer is \( 9 \times 10^6 \times 5 = 45 \times 10^6 \).

**Number of numbers satisfying condition 1, 2 and complement of 3**
Now, since the digit 6 cannot appear anywhere (due to complement of condition 3), first digit has 8 options and last digit has 4 options. Remaining 6 digits have 9 options each. So, answer is \( 8 \times 9^6 \times 5 = 32 \times 9^6 \).

So, the final answer is \( 45 \times 10^6 - 32 \times 9^6 \). ■

3 Give the closed form expression for \( \sum_{k=0}^{n} \binom{n}{k} (\frac{2}{3})^k \).
Recall the binomial formula:

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} (x)^k\]

The summation in the question resembles RHS of binomial formula with \(x = \frac{2}{3}\). So, the answer is:

\[\left(1 + \frac{2}{3}\right)^n = \left(\frac{5}{3}\right)^n\]

4. Solve the following recurrences: (assume \(T(0) = T(1) = 1\))

(a) \(T(n) = 2T(n/2) + 5\). (Assume \(n\) is a power of 2).

(b) \(T(n) = T(n - 1) + 15\)

(c) \(T(n) = T(n/2) + n\) (Assume \(n\) is a power of 2).

(d) \(T(n) = T(n - 1) + n\)

(e) \(T(n) = 2T(n - 1)\)

Part (a)

Since we can assume \(n\) is a power of 2, let \(n = 2^k\) for some \(k\). To unroll the recurrence relation,

\[T(n) = 2T(n/2) + 5\]

since we need to substitute for \(T(n/2)\), replace \(n\) with \(n/2\) and \(n/4\) in the above equation:

\[\begin{align*}
T(n/2) &= 2T(n/4) + 5 \\
T(n/4) &= 2T(n/8) + 5
\end{align*}\] (1) (2)

Hence, we get:

\[\begin{align*}
T(n) &= 2T(n/2) + 5 \\
&= 2 \times (2T(n/4) + 5) + 5 \text{ (from eqn (1))} \\
&= 4 \times T(n/4) + 2 \times 5 + 5 \\
&= 4 \times (2T(n/8) + 5) + 2 \times 5 + 5 \text{ (from eqn (2))} \\
&= 8 \times T(n/8) + 4 \times 5 + 2 \times 5 + 5
\end{align*}\]
And, continuing the pattern, after \( k \) steps, we get,

\[
T(n) = 2^k \times T(n/2^k) + 2^{k-1} \times 5 + 2^{k-2} \times 5 + \ldots + 2^0 \times 5
\]

\[
= 2^k \times T(1) + [2^{k-1} + \ldots + 1] \times 5 \quad \text{(because } n = 2^k) 
\]

\[
= 2^k \times T(1) + [2^k - 1] \times 5 \quad \text{(sum of Geometric Progression)}
\]

\[
= 6 \times 2^k - 5 \quad \text{(because } T(n) = 1) 
\]

\[
= 6n - 5 \quad \text{(because } n = 2^k)
\]

Now, let us verify the solution obtained. \( T(1) = 6 \times 1 = 5 \).

For recurrence relation, \( T(n/2) = 6 \times (n/2) - 5 = 3n - 5 \). Now, from question, we have:

\[
T(n) = 2T(n/2) + 5
\]

\[
= 2(3n - 5) + 5
\]

\[
= 6n - 5
\]

And, this verifies the solution.

\[\square\]

Part (b)
We have

\[
T(n) = T(n - 1) + 15
\]

In the above equation, by replacing \( n \) with \( n - 1, n - 2 \) and so on, we get:

\[
T(n) = T(n - 1) + 15
\]

\[
= (T(n - 2) + 15) + 15
\]

\[
= (T(n - 3) + 15) + 15 + 15
\]

\[
= (T(n - 4) + 15) + 15 + 15 + 15
\]

Finally, we get,

\[
T(n) = T(0) + 15n = 15n + 1
\]
To verify the solution, for $n = 1$, $T(0) = 15 \times 0 + 1 = 1$. Now, assuming $T(n - 1) = 15(n - 1) + 1$, we get:

$$T(n) = T(n - 1) + 15$$
$$= 15(n - 1) + 1 + 15$$
$$= 15n + 1$$

Part (c)

$$T(n) = T(n/2) + n$$

This question is very similar to Part (a). Assume $n = 2^k$ for some $k$. Following the same pattern as Part (a), we have:

$$T(n) = T(n/2) + n$$
$$= T(n/4) + n/2 + n$$
$$= T(n/8) + n/4 + n/2 + n$$

Continuing for $k$ steps, we get:

$$T(n) = T(n/2^k) + n/2^{k-1} + \ldots + n/4 + n/2 + n$$
$$= 1 + 2 + 2^2 + \ldots + 2^{k-1} + 2^k \text{ (because } n = 2^k)$$
$$= 2^{k+1} - 1$$
$$= 2n - 1$$

To verify, $T(1) = 2 \times 1 - 1 = 1$. Now, assume $T(n/2) = 2(n/2) - 1$. From recurrence relation, we have:

$$T(n) = T(n/2) + n$$
$$= 2(n/2) - 1 + n$$
$$= 2n - 1$$
Part (d)

\[ T(n) = T(n - 1) + n \]

This was solved in class. This is essentially, sum of first \( n \) integers. The answer is

\[ T(n) = \frac{n(n + 1)}{2} \]

Part (e)

\[ T(n) = 2T(n - 1) \]

For this, let us use the same method to unroll the recurrence.

\[
T(n) = 2 \times T(n - 1) \\
= 2(2 \times T(n - 2)) \\
= 2 \times 2 \times (2 \times T(n - 3))
\]

Finally, we get:

\[
T(n) = 2^n \times T(0) \\
= 2^n
\]

Now, let us verify \( T(n) = 2^n \). For \( n = 0 \), \( T(0) = 2^0 = 1 \). Assume \( T(n - 1) = 2^{n-1} \). From recurrence, we get:

\[
T(n) = 2 \times T(n - 1) \\
= 2 \times 2^{n-1} \\
= 2^n
\]

This verifies the solution.