1. (15 Marks) How many ways can you divide \( n \) identical chocolates among \( k \) children such that each gets at least 2?

**Answer:** First we give 2 chocolates to every child. Thus we are left with \( n - 2k \) chocolates. We then divide the chocolates among \( k \) children in anyway (where a child may not even get a chocolate). Note that since the children are all distinguishable so we can do it \( \binom{n-2k+k-1}{k-1} = \binom{n-k-1}{k-1} \) ways.

2. (20 Marks) How many functions \( f \) are there from \( \{1, \ldots, n\} \) to \( \{-1, 0, 1\} \) such that there exists at least one \( i \in \{1, \ldots, n\} \) where \( f(i) = 0 \).

**Answer:** The total number of functions from \( \{1, \ldots, n\} \) to \( \{-1, 0, 1\} \) is \( 3^n \). The total number of functions such that there is no \( i \) where \( f(i) = 0 \) is same as the the total number of functions from \( \{1, \ldots, n\} \) to \( \{-1, 1\} \) which is \( 2^n \). So the total number of functions from \( \{1, \ldots, n\} \) to \( \{-1, 0, 1\} \) such that there exists at least one \( i \in \{1, \ldots, n\} \) where \( f(i) = 0 \) is \( 3^n - 2^n \).
3. (20 Marks) If \( T(n) = T(n/3) + 15 \) and \( T(1) = 0 \) then guess the expression of \( T(n) \). (Assume that \( T(n) \) is a power of 3.)

**Answer:** \( T(n) = T(n/3) + 15 \). Now opening up \( T(n/3) \) we get:
\[
T(n) = (T(n/9) + 15) + 15 = T(n/3^2) + 2 \times 15
\]
Opening up \( T(n/9) \) we get
\[
T(n) = T(n/3^3) + 3 \times 15.
\]
Thus we can guess that if we continue like this we will have
\[
T(n) = T(n/3^k) + k \times 15. \tag{1}
\]
Now let \( 3^k = n \) that is \( k = \log_3 n \). Plugging it in Equation 1 we have
\[
T(n) = T(1) + 15 \log_3 n = 15 \log_3 n,
\]
the last equality because \( T(1) = 0 \).

4. (15 Marks) Give a closed form expression of \( 1 + \binom{n}{1}2 + \binom{n}{2}4 + \cdots + \binom{n}{n}2^n \).

**Answer:** By binomial theorem we can see that the above expression is same as \((1 + 2)^n\) which is \(3^n\).
5. (15 Marks) If $T(n) = T(n-1) + n$ prove that $T(n) = n(n+1)/2$.

Note: here I forgot to give that $T(0) = 0$.

**Answer:** We prove this by induction on $n$.

Base Case: $n = 0$. Clearly $T(0) = 0 = n(n-1)/2$.

Induction Hypothesis: for all $n < k$ let $T(n) = n(n+1)/2$.

Inductive Step: Now consider the case of $n = k + 1$. By the given recurrence we have $T(k+1) = T(k) + (k + 1)$. By induction hypothesis $T(k) = k(k+1)/2$. So we have

$$T(k + 1) = k(k + 1)/2 + (k + 1) = (k + 1)(k + 2)/2 = (k + 1)((k + 1) + 1)/2.$$ 

So we prove that for all $n \geq 0$, $T(n) = n(n+1)/2$.

6. (15 Marks) Prove the following statement: $2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)$

**Answer:** We prove this by induction on $n$.

Let $T(n) = 2^2 + \cdots + (3n-1)^2$. So to prove: $T(n) = \frac{1}{2}n(6n^2 + 3n - 1)$

Base Case: $n = 0$. Clearly $T(1) = 4$ and $\frac{1}{2}1(6*1^2 + 3*1 - 1) = 4$. Thus base case is proved.

Induction Hypothesis: for all $n < k$ let $T(n) = \frac{1}{2}n(6n^2 + 3n - 1)$.

Inductive Step: Now consider the case of $n = k + 1$. By the given recurrence we have

$$T(k + 1) = T(k) + (3(k + 1) - 1)^2 = T(k) + (9k^2 + 12k + 4).$$

By induction hypothesis $T(k) = \frac{1}{2}k(6k^2 + 3k - 1)$. So we have

$$T(k + 1) = \frac{1}{2}k(6k^2 + 3k - 1) + (9k^2 + 12k + 4).$$

If we solve the last expression we see it is same as $\frac{1}{2}(k + 1)(6(k + 1)^2 + 3(k + 1) - 1)$.

So we prove that for all $n \geq 1$, $T(n) = \frac{1}{2}n(6n^2 + 3n - 1)$. 

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7. (15 Marks) Prove that for all integer \( n \geq 3 \), \( n^2 - 7n + 12 \geq 0 \).

**Answer:** We prove this by induction on \( n \).

**Base Case:** \( n = 3 \). Clearly \( n^2 - 7n + 12 = 0 \geq 0 \).

**Induction Hypothesis:** for all \( n < k \) let \( n^2 - 7n + 12 \geq 0 \).

**Inductive Step:** Now consider the case of \( n = k + 1 \). Now

\[
(k + 1)^2 - 7(k + 1) + 12 = k^2 + 5k + 7 = (k^2 - 7k + 12) + (2k - 5) \quad (2)
\]

By induction hypothesis \( k^2 + 7k + 12 \geq 0 \). Also for all \( k \geq 3 \) we have \((2k - 5) \geq 1 \geq 0 \). Thus we have from Equation 2

\[
(k + 1)^2 - 7(k + 1) + 12 \geq 0
\]

So we prove that for all \( n \geq 3 \), \((k + 1)^2 - 7(k + 1) + 12 \geq 0 \).