1. Let \( x > -1 \) be a real number. Prove that \( (1 + x)^n \geq 1 + nx \) for all natural numbers \( n \).

**Proof.** By induction on \( n \).

**Base Case:** For \( n = 1 \), the lefthand side of the inequality is \( (1 + x)^1 = 1 + x \) and the righthand side is \( 1 + 1 \cdot x = 1 + x \). \( 1 + x \geq 1 + x \), so the inequality is true for the base case.

**Inductive Hypothesis:** Assume that for some natural number \( k \) \( (1 + x)^k \geq 1 + kx \) holds.

**Inductive Step:** We want to show that \( (1 + x)^{k+1} \geq 1 + (k+1)x \) is also true. We start with the inductive hypothesis:

\[
\begin{align*}
(1 + x)^k & \geq 1 + kx & \text{by inductive hypothesis} \\
(1 + x)^k(1 + x) & \geq (1 + kx)(1 + x) & \text{multiplying } (1 + x) \text{ on both sides} \\
(1 + x)^{k+1} & \geq 1 + x + kx + kx^2 \\
(1 + x)^{k+1} & \geq 1 + (k+1)x + kx^2 & \text{factoring } x \text{ on right side}
\end{align*}
\]

We know that \( k \) is positive and \( x^2 \) must either be 0 or positive because it is a square. Thus, \( kx^2 \geq 0 \). This means that \( 1 + (k+1)x + kx^2 \geq 0 \). By transitivity, \( (1 + x)^{k+1} \geq 1 + (k+1)x \), as required. \( \Box \)

2. Prove that \( \sum_{i=1}^{n} i \times i! = (n+1)! - 1 \), for all positive integers \( n \).

**Proof.** By induction on \( n \).

**Base Case:** For \( n = 1 \) the lefthand side of the equation is \( \sum_{i=1}^{1} i \times i! = 1 \times 1! = 1 \). The righthand side of the equation is \( (1 + 1)! - 1 = 2! - 1 = 1 \). Since \( 1 = 1 \), the statement holds for the base case.

**Inductive Hypothesis:** Assume for some positive integer \( k \), \( \sum_{i=1}^{k} i \times i! = (k+1)! - 1 \).

**Inductive Step:** We want to show that \( \sum_{i=1}^{k+1} i \times i! = (k+2)! - 1 \). We start with the inductive hypothesis:

\[
\begin{align*}
\sum_{i=1}^{k} i \times i! & = (k+1)! - 1 & \text{by inductive hypothesis} \\
\sum_{i=1}^{k} i \times i! + (k+1)(k+1)! & = (k+1)! - 1 + (k+1)(k+1)! & \text{add } (k+1)(k+1)! \text{ to both sides} \\
\sum_{i=1}^{k+1} i \times i! & = (k+1)!(1 + (k + 1)) - 1 & \text{factor } (k + 1)! \text{ on right side}
\end{align*}
\]
By simplifying the right side of the equation we get \( \sum_{i=1}^{k+1} i \times i! = (k + 2)! - 1 \) as required.

3. Let \( \{a_n\} \) be a sequence of natural numbers such that \( a_1 = 5 \), \( a_2 = 13 \) and \( a_{n+2} = 5a_{n+1} - 6a_n \) for all natural numbers \( n \). Prove that \( a_n = 2^n + 3^n \) for all natural number \( n \).

**Proof.** By strong induction on \( n \).

**Base cases:** For \( n = 1 \), \( a_1 = 2^1 + 3^1 = 5 \), which is true. For \( n = 2 \), \( a_2 = 2^2 + 3^2 = 13 \), which is also true.

**Inductive Hypothesis:** Assume for all natural numbers \( m \leq k - 1 \), \( a_m = 2^m + 3^m \)

**Inductive Step:** We want to show that \( a_k = 2^k + 3^k \). We start with the recurrence relation:

\[
a_k = 5a_{k-1} - 6a_{k-2}
\]

\[
= 5(2^{k-1} + 3^{k-1}) - 6(2^{k-2} + 3^{k-2})
\]

\[
= 5 \cdot 2^{k-1} + 5 \cdot 3^{k-1} - 6 \cdot 2^{k-2} - 6 \cdot 3^{k-2}
\]

\[
= 2^{k-2}(5 \cdot 2 - 6) + 3^{k-2}(5 \cdot 3 - 6)
\]

\[
= 2^{k-2} \cdot 4 + 3^{k-2} \cdot 9
\]

\[
= 2^{k-2} \cdot 2^2 + 3^{k-2} \cdot 3^2
\]

\[
= 2^k + 3^k
\]

as required.

4. Prove that for all \( n \geq 1 \),

\[
1^3 + 2^3 + \cdots + n^3 = \left( \frac{n(n+1)}{2} \right)^2.
\]

**Proof.** By induction on \( n \).

**Base case:** For \( n = 1 \), \( 1^3 = 1 \), \( \left( \frac{1(1+1)}{2} \right)^2 = 1 \). \( 1 = 1 \) so the statement holds for the base case.

**Inductive Hypothesis:** Assume for some \( k \geq 1 \), \( 1^3 + 2^3 + \cdots + k^3 = \left( \frac{k(k+1)}{2} \right)^2 \)

**Inductive Step:** We want to show that \( 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2 \). We start with the inductive hypothesis:
\[1^3 + 2^3 + \cdots + k^3 = \left( \frac{k(k+1)}{2} \right)^2 \]  
by inductive hypothesis

\[1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3 \]  
add \((k+1)^3\) to both sides

\[= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \]

\[= \frac{(k+1)^2(k^2+4(k+1))}{4} \]  
factoring \((k+1)^2\) on numerator

\[= \frac{(k+1)^2(k^2+4k+4)}{4} \]

By simplifying the righthand side we get \(1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2\), as required. 

5. For all \(n > 4\) prove that \(3^n > 7(2^n)\).

\textbf{Proof.} By induction on \(n\).

Base case: For \(n = 5\).  \(3^5 = 243\) and \(7(2^5) = 224\).  \(243 > 224\), so the statement holds for the base case.

Inductive Hypothesis: Assume for some \(k > 4\), \(3^k > 7(2^k)\)

Inductive Step: We want to show that \(3^{k+1} > 7(2^{k+1})\). We start with the inductive hypothesis:

\[3^k > 7(2^k) \]  
by inductive hypothesis

\[3^k \cdot 3 > 7(2^k) \cdot 3 \]  
multiplying 3 on both sides

\[7(2^k) \cdot 3 > 7(2^k) \cdot 2 \]  
because \(3 > 2\)

So by transitivity and simplification, we get \(3^{k+1} > 7(2^{k+1})\), as required. 

\(\square\)