CSE 20: Assignment Set 4

1. Prove that proving \((A \implies B)\) is same as proving \((\neg B \implies \neg A)\).

Using a truth table:

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>\neg B</th>
<th>\neg A</th>
<th>A \implies B</th>
<th>\neg B \implies \neg A</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
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</tr>
</tbody>
</table>

The last two columns are the same, thus \(A \implies B\) is equivalent to \((\neg B \implies \neg A)\).

2. Is the statement form \((p \land q) \lor (\neg p \lor (p \land \neg q))\) a tautology or a contradiction or none.

Answer: The statement is a tautology.

Method 1: Truth Table

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>\neg q</th>
<th>p \land \neg q</th>
<th>\neg p</th>
<th>\neg p \lor (p \land \neg q)</th>
<th>p \land q</th>
<th>(p \land q) \lor (\neg p \lor (p \land \neg q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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</tr>
</tbody>
</table>

The last column shows that the statement is a tautology.

Method 2: Algebraic Manipulations

\[(p \land q) \lor (\neg p \lor (p \land \neg q))\]
\[(p \land q) \lor ((\neg p \lor p) \land (\neg p \lor \neg q))\] distributive rule
\[(p \land q) \lor (T \lor (\neg p \lor \neg q))\] negation rule
\[(p \land q) \lor (\neg p \lor \neg q)\] bound rule
\[(p \land q) \lor \neg(p \land q)\] DeMorgan’s rule
\[T\] negation rule

Thus, the statement is a tautology.

3. Let \(D = \{-48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36\}\). Determine which of the following statements are true and which are false. Provide counterexamples for those statements that are false. Prove the statements that are true.
(a) \( \forall x \in D, \) if \( x \) is odd then \( x > 0. \)
(b) \( \forall x \in D, \) if \( x \) is less than 0 then \( x \) is even.
(c) \( \forall x \in D, \) if \( x \) is even then \( x \leq 0. \)
(d) \( \forall x \in D, \) if the ones digit of \( x \) is 6, then the tens digit is 1 or 2.

(a) **Answer:** True.
All odd numbers in \( D \) are positive.
(b) **Answer:** True.
All negative numbers in \( D \) are even.
(c) **Answer:** False.
Counterexamples: 16, 26, 32, 36
(d) **Answer:** False.
Counterexample: 36

4. Write a negation for each statement. Bring the negation as deeply into the statement as possible.

(a) For all real numbers \( x, \) if \( x^2 \geq 1 \) then \( x > 0. \)
(b) For all integers \( d, \) if \( \frac{6}{d} \) is an integer then \( d = 3. \)
(c) For all real numbers \( x, \) if \( x(x+1) > 0 \) then \( x > 0 \) or \( x < -1. \)
(d) For all integers \( a, b, \) and \( c, \) if \( a - b \) is even and \( b - c \) is even then \( a - c \) is even.

(a) **Answer:** There exists a real number \( x \) such that \( x^2 \geq 1 \) and \( x \leq 0. \)

Statement: \( \forall x \in \mathbb{R} \ (x^2 \geq 1 \implies x > 0) \)
Negation: \( \neg(\forall x \in \mathbb{R} \ (x^2 \geq 1 \implies x > 0)) \)
\( \exists x \in \mathbb{R} \ (x^2 \geq 1 \land \neg(x > 0)) \)
\( \exists x \in \mathbb{R} \ (x^2 \geq 1 \land x \leq 0) \)

(b) **Answer:** There exists an integer \( d \) such that \( \frac{6}{d} \) is an integer and \( d \neq 3. \)

Statement: \( \forall d \in \mathbb{Z} \ (\frac{6}{d} \in \mathbb{Z} \implies d = 3) \)
Negation: \( \neg(\forall d \in \mathbb{Z} \ (\frac{6}{d} \in \mathbb{Z} \implies d = 3)) \)
\( \exists d \in \mathbb{Z} \ (\frac{6}{d} \in \mathbb{Z} \land \neg(d = 3)) \)
\( \exists d \in \mathbb{Z} \ (\frac{6}{d} \in \mathbb{Z} \land d \neq 3) \)

(c) **Answer:** There exists a real number \( x \) such that \( x(x+1) > 0, \) \( x \leq 0, \) and \( x \geq -1. \)

Statement: \( \forall x \in \mathbb{R} \ (x(x+1) > 0 \implies (x > 0 \lor x < -1)) \)
Negation: \( \neg(\forall x \in \mathbb{R} \ (x(x+1) > 0 \implies (x > 0 \lor x < -1))) \)
\( \exists x \in \mathbb{R} \ (\neg((x(x+1) > 0 \implies (x > 0 \lor x < -1)))) \)
\( \exists x \in \mathbb{R} \ ((x(x+1) > 0 \land \neg(x > 0 \lor x < -1))) \)
\( \exists x \in \mathbb{R} \ ((x(x+1) > 0 \land (x \leq 0 \land x \geq -1))) \)

(d) **Answer:** There exist 3 integers \( a, b, \) and \( c \) such that \( a - b \) is even, \( b - c \) is even, and \( a - c \) is odd.
5. Let $x$ be an integer. Prove that if $x^2 - 6x + 5$ is even then $x$ must be odd.

Proof. Assume for the sake of contradiction that $x^2 - 6x + 5$ is even and $x$ is even. By definition of even, $x = 2c$ where $c$ is an integer. By substitution, $x^2 - 6x + 5 = (2c)^2 - 6(2c) + 5 = 4c^2 - 12c + 5 = 4c^2 - 12c + 4 + 1 = 2(2c^2 - 6c + 2) + 1$. Since $2c^2 - 6c + 2$ is an integer, $x^2 - 5x + 5$ must be odd, which is a contradiction. \qed