1. Let \( r = "\text{she registered to vote}" \) and \( v = "\text{she voted}" \). Write the following statement in symbolic form: She registered to vote but she did not vote.

"She registered to vote" AND "she did not vote"
"She registered to vote" AND NOT ("she voted")

**Answer:** \( r \land \sim v \)

2. Make a truth table for \((p \lor (\sim p \lor q)) \land (q \land \sim r)\)

I will show two ways to do this problem. The first way involves only filling a truth table. The second way involves algebraic manipulations before making the truth table.

**Method 1**

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<th>( \sim p )</th>
<th>( \sim p \lor q )</th>
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<th>( q \land \sim r )</th>
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**Method 2**

*Note:* This method uses the algebraic rules for boolean functions (Theorem 2 BF-6 of textbook). I tried to be as clear as possible by writing out every rule that was used in separate steps.

\[(p \lor (\sim p \lor q)) \land (q \land \sim r)\]

\[
\begin{align*}
(p \lor (\sim p \lor q)) & \sim (q \land \sim r) \\
((p \lor \sim p) \lor q) & \sim (q \land \sim r) \\
(T \lor q) & \sim (q \land \sim r) \\
T & \sim (q \land \sim r) \\
T \land (\sim q \lor \sim r) & \text{DeMorgan’s rule} \\
T \land (\sim q \lor r) & \text{double negation} \\
\sim q \lor r & \text{bound rule}
\end{align*}
\]

This means that the truth table for \((p \lor (\sim p \lor q)) \land (q \land \sim r)\) is equivalent to \(\sim q \lor r\) so we can make a much simpler table.
3. Using DeMorgan’s rule, state the negation of the statement: “The car is out of gas or the fuel line is plugged.”

Let \( p = \) “the car is out of gas” and \( q = \) “the fuel line is plugged.” Then, the statement \( s = p \lor q \). The negation of \( s \) is \( \sim s = \sim (p \lor q) = \sim p \land \sim q \) by DeMorgan’s rule. So the negation reads “The car is not out of gas and the fuel line is not plugged.”

4. A pair of numbers \( x \) and \( y \) satisfy a system of inequalities if

\[
\begin{align*}
3 &\leq x \leq 5 \\
|x - y| &< 1.
\end{align*}
\]

What are the conditions under which \( x \) and \( y \) fail to satisfy this system?

This system can also be written as a conjunction of two statements. Let \( p = 3 \leq x \leq 5 \) and \( q = |x - y| < 1 \). Then \( \sim (p \land q) = \sim p \lor \sim q \) by DeMorgan’s rule. So system fails when \( x < 3 \) or \( x > 5 \) or \( |x - 1| \geq 1 \)

5. Is the function \( (p \land (\sim (p \lor q))) \lor (p \land q) \) equal to the function \( p \lor q \)? Why or why not?

**Answer:** No.

Like question 2, I will use two methods to answer this question. The first way is uses truth tables and the second way uses the algebraic rules.

**Method 1**

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
p & q & \sim p & \sim p \lor q & \sim (p \lor q) & p \land (\sim (p \lor q)) & p \land q & (p \land (\sim (p \lor q))) \lor (p \land q) \\
\hline
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\hline
\end{array}
\]

The last two columns of the table are not the same, and thus, the two statements are not equivalent.

**Method 2**
6. If \( a \) and \( b \) are two positive integers then prove that \( a^2 - 4b \) cannot be equal to 2. (Hint: Prove using contradiction.)

Proof. Suppose for contradiction that \( a \) and \( b \) are positive integers and that \( a^2 - 4b = 2 \). Subtracting 4\( b \) from both sides gives us \( a^2 = 2 + 4b = 2(1 + 4b) \). Since 1 + 4\( b \) is an integer, \( a^2 \) must be even, which means that \( a \) must be even. So, we can write \( a = 2c \), where \( c \) is an integer. Substituting 2\( c \) for \( a \), we get

\[
(2c)^2 - 4b = 2 \\
4c^2 - 4b = 2 \\
2c^2 - 2b = 1 \\
2(c^2 - b) = 1
\]

The lefthand side of this equation is even, but the righthand side is odd, which is a contradiction. \( \square \)