CSE 20
Lecture 8: Propositional Logic (contd...)
Outline of the course

Quizes and Midterm:

- Quiz 2: on 3rd February
- Midterm: on 14th February
- Quiz 3: on 26th February
- Quiz 4: on 5th March
- Quiz 5: on 12th March
Part 3a: Show that if \( a \) and \( b \) are integers in the range 1 through 256, and the sum of \( a \) and \( b \) is also in this range, then

\[
2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.
\]

Part 3b: Note: \( 2^9 = 512 \). Explain why it follows that the binary representation of \((2^9 - a) + (2^9 - b)\) has a leading term in the \(2^9\)th position.
Show that if $a$ and $b$ are integers in the range 1 through 256, and the sum of $a$ and $b$ is also in this range, then

$$2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.$$
Rephrasing the question

Show that if \( a \) and \( b \) are integers in the range 1 through 256, and the sum of \( a \) and \( b \) is also in this range, then

\[
2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.
\]

Show that if

- \( a \) is an integers in the range 1 through 256, \ AND \
Rephrasing the question

Show that if $a$ and $b$ are integers in the range 1 through 256, and the sum of $a$ and $b$ is also in this range, then

$$2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.$$  

Show that if

- $a$ is an integers in the range 1 through 256, AND
- $b$ is an integers in the range 1 through 256, AND
Rephrasing the question

Show that if $a$ and $b$ are integers in the range 1 through 256, and the sum of $a$ and $b$ is also in this range, then

$$2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.$$ 

Show that if

- $a$ is an integers in the range 1 through 256, AND
- $b$ is an integers in the range 1 through 256, AND
- the sum of $a$ and $b$ is in the range 1 through 256,
Rephrasing the question

Show that if $a$ and $b$ are integers in the range 1 through 256, and the sum of $a$ and $b$ is also in this range, then

$$2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.$$ 

Show that if

- $a$ is an integers in the range 1 through 256, AND
- $b$ is an integers in the range 1 through 256, AND
- the sum of $a$ and $b$ is in the range 1 through 256,

IMPLIES
Rephrasing the question

Show that if \( a \) and \( b \) are integers in the range 1 through 256, and the sum of \( a \) and \( b \) is also in this range, then

\[
2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.
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Show that if

- \( a \) is an integers in the range 1 through 256, AND
- \( b \) is an integers in the range 1 through 256, AND
- the sum of \( a \) and \( b \) is in the range 1 through 256,

IMPLIES \( 2^9 \leq (2^9 - a) + (2^9 - b) \), AND
Rephrasing the question

Show that if $a$ and $b$ are integers in the range 1 through 256, and the sum of $a$ and $b$ is also in this range, then

$$2^9 \leq (2^9 - a) + (2^9 - b) < 2^{10}.$$ 

Show that if

- $a$ is an integer in the range 1 through 256, AND
- $b$ is an integer in the range 1 through 256, AND
- the sum of $a$ and $b$ is in the range 1 through 256,

IMPLIES $2^9 \leq (2^9 - a) + (2^9 - b)$, AND $(2^9 - a) + (2^9 - b) < 2^{10}$. 
• $a$ is an integer in the range 1 through 256, ($p$) AND
• $b$ is an integer in the range 1 through 256, ($q$) AND
• the sum of $a$ and $b$ is in the range 1 through 256, ($r$)

IMPLIES

\[ 2^9 \leq (2^9 - a) + (2^9 - b), \quad (s) \quad \text{AND} \]
\[ (2^9 - a) + (2^9 - b) < 2^{10}. \quad (t) \]

To Prove:

\[ (p \land q \land r) \implies (s \land t) \]
How to prove the problem

To Prove: \((p \land q \land r) \implies (s \land t)\)
How to prove the problem

To Prove: \((p \land q \land r) \implies (s \land t)\)
Notice that:

Similarly, \((p \land q) \implies (s \land t)\) IMPLIES \((p \land q \land r) \implies (s \land t)\).
To Prove: \((p \land q \land r) \implies (s \land t)\)

Notice that:

\[(r \implies s) \land (r \implies t)\]

**IMPLIES**

\[(p \land q \land r) \implies (s \land t)\]
How to prove the problem

To Prove: \((p \land q \land r) \implies (s \land t)\)

Notice that:

\((r \implies s) \land (r \implies t)\)

IMPLIES

\((p \land q \land r) \implies (s \land t)\)

Similarly,

\((p \land q) \implies (s \land t)\) IMPLIES \((p \land q \land r) \implies (s \land t)\).
Show that if

- $a$ is an integer in the range 1 through 256, (\(\equiv p\)) AND
- $b$ is an integer in the range 1 through 256, (\(\equiv q\)) AND
- the sum of $a$ and $b$ is in the range 1 through 256, (\(\equiv r\))

IMPLIES

\[2^9 \leq (2^9 - a) + (2^9 - b), \ (\equiv s) \ AND\]
\[(2^9 - a) + (2^9 - b) < 2^{10}. \ (\equiv t)\]

To Prove:

\[(r \implies s) \land (r \implies t)\]
Direct proof of the statement

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, (=r)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (\equiv s) \text{ AND } (2^9 - a) + (2^9 - b) < 2^{10}. (= t)$$
Direct proof of the statement

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($=r$)

IMPLIES

$2^9 \leq (2^9 - a) + (2^9 - b)$, ($=s$) AND

$(2^9 - a) + (2^9 - b) < 2^{10}$. ($=t$)

To Prove: $(r \implies s) \land (r \implies t)$
Direct proof of the statement

Show that if

- the sum of \( a \) and \( b \) is in the range 1 through 256, \( (=r) \)

IMPLIES

\[
2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s) \quad \text{AND} \\
(2^9 - a) + (2^9 - b) < 2^{10}. (= t)
\]

To Prove: \( (r \implies s) \land (r \implies t) \)

\[
(2^9 - a) + (2^9 - b) = 2^{10} - (a + b) < 2^{10} \quad \text{[Since } (a + b) \geq 1]\]
Direct proof of the statement

Show that if

• the sum of $a$ and $b$ is in the range 1 through 256, ($=r$)

**IMPLIES**

\[ 2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s) \text{ AND} \]
\[ (2^9 - a) + (2^9 - b) < 2^{10}. (= t) \]

To Prove: $(r \implies s) \land (r \implies t)$

\[ (2^9 - a) + (2^9 - b) = 2^{10} - (a + b) < 2^{10} \quad [\text{Since } (a + b) \geq 1] \]
\[ (2^9 - a) + (2^9 - b) = 2^9 + (2^9 - (a + b)) \geq 2^9 \]

[Since $2^9 \geq (a + b)$]
To prove statement $B$ from $A$:
Proof Techniques

To prove statement \( B \) from \( A \):

- **Direct Proof:**
  \[ A \implies B \]
Proof Techniques

To prove statement $B$ from $A$:

- **Direct Proof:**
  \[ A \implies B \]

- **Proof by contradiction:**
  \[ (\neg B \land A) \text{ gives a contradiction} \]
Proof of the statement by contradiction

Show that if
- the sum of \( a \) and \( b \) is in the range 1 through 256, (=r)

\[
2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s)
\]

To Prove: \((\neg s \land r)\) gives a contradiction
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($=r$)

implies

$$2^9 \leq (2^9 - a) + (2^9 - b), \ (=s)$$

To Prove: ($\neg s \land r$) gives a contradiction
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, (=r)

\[ 2^9 \leq (2^9 - a) + (2^9 - b), \text{ (}=s) \]

To Prove: $(\neg s \land r)$ gives a contradiction
Proof of the statement by contradiction

Show that if

1. the sum of $a$ and $b$ is in the range 1 through 256, ($=r$)

   \[
   2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s)
   \]

To Prove: $(\neg s \land r)$ gives a contradiction

For proving the above statement using contradiction what do we have to prove?
Proof of the statement by contradiction

Show that if the sum of $a$ and $b$ is in the range 1 through 256 THEN $2^9 \leq (2^9 - a) + (2^9 - b)$.

For proving the above statement using contradiction what do we have to prove?

1. If $2^9 \leq (2^9 - a) + (2^9 - b)$ then sum of $a$ and $b$ is in the range 1 through 256
2. If $2^9 < (2^9 - a) + (2^9 - b)$ then sum of $a$ and $b$ is in the range 1 through 256
3. If $2^9 > (2^9 - a) + (2^9 - b)$ AND the sum of $a$ and $b$ is in the range 1 through 256 then there is a problem
4. If $2^9 \geq (2^9 - a) + (2^9 - b)$ AND the sum of $a$ and $b$ is in the range 1 through 256 then there is no problem
Proof of the statement by contradiction

Show that if the sum of $a$ and $b$ is in the range 1 through 256 THEN $2^9 \leq (2^9 - a) + (2^9 - b)$.

For proving the above statement using contradiction what do we have to prove?

1. If $2^9 \leq (2^9 - a) + (2^9 - b)$ then sum of $a$ and $b$ is in the range 1 through 256
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4. If $2^9 \geq (2^9 - a) + (2^9 - b)$ AND the sum of $a$ and $b$ is in the range 1 through 256 then there is NO PROBLEM
Proof of the statement by contradiction

Show that if the sum of $a$ and $b$ is in the range 1 through 256 THEN \( 2^9 \leq (2^9 - a) + (2^9 - b) \).

For proving the above statement using contradiction what do we have to prove?

1. If \( 2^9 \leq (2^9 - a) + (2^9 - b) \) then sum of $a$ and $b$ is in the range 1 through 256
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4. If \( 2^9 \geq (2^9 - a) + (2^9 - b) \) AND the sum of $a$ and $b$ is in the range 1 through 256 then there is NO PROBLEM
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($\Rightarrow r$)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (\Rightarrow s)$$
Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($\equiv r$)

**IMPLIES**

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (\equiv s)$$

To Prove: ($\neg s \land r$) gives a contradiction
Proof of the statement by contradiction

Show that if

- the sum of \( a \) and \( b \) is in the range 1 through 256, \((=r)\)

\[
2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s)
\]

**To Prove:** \((\neg s \land r)\) gives a contradiction

\[
2^9 > (2^9 - a) + (2^9 - b)
\]
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($\equiv r$)

**IMPLIES**

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (\equiv s)$$

To Prove: $(\neg s \land r)$ gives a contradiction

$$2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)$$
Proof of the statement by contradiction

Show that if

- the sum of \(a\) and \(b\) is in the range 1 through 256, \((\equiv r)\)

\[
2^9 \leq (2^9 - a) + (2^9 - b), \ (\equiv s)
\]

To Prove: \((\neg s \land r)\) gives a contradiction

\[
2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)
\]

\[
\implies (a + b) > 2^{10} - 2^9
\]
Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($=r$)

**IMPLIES**

$$2^9 \leq (2^9 - a) + (2^9 - b), \ (=s)$$

To Prove: ($\neg s \land r$) gives a contradiction

$$2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)$$

$$\implies (a + b) > 2^{10} - 2^9 = 2^9$$
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($\equiv r$)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (\equiv s)$$

To Prove: ($\neg s \land r$) gives a contradiction

$$2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)$$
$$\implies (a + b) > 2^{10} - 2^9 = 2^9$$
$$\implies 2^9 = 512 < (a + b)$$
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($=r$)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \ (=s)$$

To Prove: $(\neg s \land r)$ gives a contradiction

$$2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)$$
$$\implies (a + b) > 2^{10} - 2^9 = 2^9$$
$$\implies 2^9 = 512 < (a + b) \leq 256 \ [\text{From } r].$$
Proof of the statement by contradiction

Show that if

- the sum of $a$ and $b$ is in the range 1 through 256, ($r$)

IMPLIES

$$2^9 \leq (2^9 - a) + (2^9 - b), \quad (=s)$$

To Prove: ($\neg s \land r$) gives a contradiction

$$2^9 > (2^9 - a) + (2^9 - b) = 2^{10} - (a + b)$$

$$\implies (a + b) > 2^{10} - 2^9 = 2^9$$

$$\implies 2^9 = 512 < (a + b) \leq 256 \quad [\text{From } r].$$

Thus, $512 < 256$ CONTRADICTION
Proof Techniques

To prove statement \( B \) from \( A \):

- **Direct Proof:** \( A \Rightarrow B \)

- **Proof by contradiction:** \((\neg B \land A)\) gives a contradiction

- **Contrapositive**

Depending on what the problem is we will have to decide which proof technique to use.
Proof Techniques

To prove statement $B$ from $A$:

- **Direct Proof:**
  \[ A \implies B \]

- **Proof by contradiction:**
  \[ (\neg B \land A) \text{ gives a contradiction} \]
Proof Techniques

To prove statement $B$ from $A$:

- **Direct Proof:**
  \[ A \implies B \]

- **Proof by contradiction:**
  \[ (\neg B \land A) \text{ gives a contradiction} \]

- **Contrapositive**
- **Induction**

Depending on what the problem is we will have to decide which proof technique to use.
Why is proof by contradiction correct?

Why is proving \( A \implies B \) same as proving

\[ (\neg B \land A) \] gives a contradiction

In other words

\( A \implies B \) is same as

\[ (\neg B \land A) = FALSE \]
Definition

Two statements are equivalent if their TRUTH TABLES are the same.
Definition

Two statements are equivalent if their TRUTHTABLES are the same.

To prove: $A \implies B$ is equivalent to $(\neg B \land A) = FALSE$
Checking Equivalence

Definition
Two statements are equivalent if their TRUTHTABLES are the same.

To prove: $A \implies B$ is equivalent to $(\neg B \land A) = \text{FALSE}$

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<th>$A \implies B$</th>
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Checking Equivalence

Definition

Two statements are equivalent if their TRUTH TABLES are the same.

To prove: \( A \implies B \) is equivalent to \( (\neg B \land A) = FALSE \)

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Checking Equivalence

Definition

Two statements are equivalent if their TRUTHTABLES are the same.

To prove: \( A \implies B \) is equivalent to \( (\neg B \land A) = FALSE \)

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Since $A \implies B$ is equivalent to $(\neg B \land A) = FALSE$ so for proving $A \implies B$ its enough to show $(\neg B \land A) = FALSE$. 
Proof of “Proof by Contradiction”

Since $A \implies B$ is equivalent to $(\neg B \land A) = FALSE$ so for proving $A \implies B$ its enough to show $(\neg B \land A) = FALSE$.

Similarly many other statements can be shown to be equivalent.

Sometimes they can be called RULES.
Rules about connective

For example: $(A \implies B)$ can be replaced to $(\neg B \land \neg A)$

In other words: $(A \implies B) \iff (\neg A \lor B)$
For example: $(A \implies B)$ can be replaced to $(\neg B \land \neg A)$

In other words: $(A \implies B) \iff (\neg A \lor B)$

How to prove this rule?
Rules about connective

For example: \((A \implies B)\) can be replaced to \((\neg B \land \neg A)\)

In other words: \((A \implies B) \iff (\neg A \lor B)\)

How to prove this rule?

TRUTH TABLE
To prove: \((A \implies B) \iff \neg A \lor B\)
Rules about connectives

To prove: \((A \Rightarrow B) \iff (\neg A \lor B)\)

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Rules about connectives

To prove: \((A \implies B) \iff (\neg A \lor B)\)

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Rules about connectives

To prove: \((A \implies B) \iff (\neg A \lor B)\)

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Equivalence of statements/propositions

If the truthtables of two statement/propositions are identical then the two statement/propositions are equivalent.

Thus:

\[ A \Rightarrow B \] is equivalent to \[ \neg A \vee B \].

Similarly:

\[ p \land (q \lor r) \iff (p \land q) \lor (p \land r) \] (Distributive rule)

\[ \neg (p \land q) \iff (\neg p \lor \neg q) \] (De Morgan’s rule)

More rules in Theorem 1 in Section 1 of Unit:Lo from Bender Williamson book.
Equivalence of statements/propositions

If the truth tables of two statements/propositions are identical then the two statements/propositions are equivalent.

Thus: $A \implies B$ is equivalent to $\neg A \lor B$.

Similarly:

- $p \land (q \lor r) \iff (p \land q) \lor (p \land r)$ (Distributive rule)
- $\neg (p \land q) \iff (\neg p \lor \neg q)$ (De Morgan’s rule)
- More rules in Theorem 1 in Section 1 of Unit:Lo from Bender Williamson book.
A statement is correct iff it is true for all inputs.

Checking a statement is correct is done using TRUTHTABLE.

Two statements are equivalent if their TRUTHTABLES are identical.

A statement which is always true is called a TAUTOLOGY.

A statement that is never true is called a CONTRADICTION.