Problem 1

For any string $w = w_1w_2\ldots w_n$, let $w^r = w_n\ldots w_2w_1$ be the reverse of $w$, i.e., $w$ written backward. The reverse of a language $L^r = \{w^r : w \in L\}$ is defined reversing each string individually. Consider the set

$$P_{TM} = \{ \langle M \rangle : M \text{ is a TM with alphabet } \{0, 1\} \text{ such that } L(M)^r = L(M) \}$$

of all Turing machines whose recognized language is closed under reverse.

(a) Give a map reduction from $A_{TM}$ to the complement of $P_{TM}$, and prove that your reduction is correct.

(b) Give a map reduction from $A_{TM}$ to $P_{TM}$, and prove that your reduction is correct.

(c) Based on your answers to part (a) and (b), answer the following questions: Is $P_{TM}$ recognizable? If $P_{TM}$ co-recognizable? You answer to each question can be “yes”, “no”, or “it does not follow from parts (a) and (b)”.

Problem 2

Prove or disprove each of the following statements:

(a) for any language $L$ other than $\emptyset$ and $\Sigma^*$, there is a map reduction from $L$ to the complement of $L$.

(b) for any language $L$ other than $\emptyset$ and $\Sigma^*$, if there is a map reduction from $L$ to the complement of $L$, then $L$ is either decidable, or neither recognizable nor co-recognizable.

(c) for any language $L$ other than $\emptyset$ and $\Sigma^*$, if $L$ is decidable, then there is a map reduction from $L$ to the complement of $L$.

Problem 3

Let $L$ be the set of all strings of the form $\langle M_1, M_2 \rangle$ where $M_1, M_2$ are Turing machines, and $L(M_1) \subseteq L(M_2)$.

(a) Determine if $L$ is recognizable or not, and prove your answer using map reductions.

(b) Determine if $L$ is co-recognizable or not, and prove your answer using map reductions.