Homework #5

[Each problem is worth 20 points. This set is challenging!]

5.1 A hand $H$ of 3 random cards are dealt from an ordinary deck of 52. Let $E_1$ denote the event that $H$ has at least 1 Ace, $E_2$ denote the event that $H$ has at least 2 Aces, and let $E_{AS}$ denote the event that $H$ includes the Ace of Spades.

(i) What are $P(E_1)$, $P(E_2)$ and $P(E_{AS})$?

(ii) What is the conditional probability $P(E_2 | E_1)$?

(iii) What is the conditional probability $P(E_2 | E_{AS})$?

(Are you surprised that the answers to (ii) and (iii) are different?)

Solution

(i) $P(E_1) = 1 - \left(\frac{48}{52}\right)^3$, $P(E_2) = \frac{\left(\frac{4}{52}\right) + \left(\frac{3}{52}\right) \left(\frac{31}{51}\right)}{\frac{3}{52} \left(\frac{2}{51}\right)}$, $P(E_{AS}) = \frac{\left(\frac{51}{52}\right)^2}{\frac{3}{52} \left(\frac{2}{51}\right)}$;

(ii) $P(E_2 | E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{P(E_2)}{P(E_1)} = \frac{\left(\frac{4}{52}\right) + \left(\frac{3}{52}\right) \left(\frac{31}{51}\right)}{\frac{3}{52} \left(\frac{2}{51}\right)}$;

(iii) $P(E_2 | E_{AS}) = 1 - \left(\frac{48}{51}\right)^2$.

5.2 An urn contains 3 Red and 4 White marbles. A fair coin is flipped. If the flip is Heads then 1 Red and 2 White marbles are added to the urn. On the other hand, if the flip is Tails, then 1 Red and 2 White marbles are removed from the urn. Two random marbles are now drawn from the urn without replacement.

(i) What is the probability that both of the drawn marbles are White?

(ii) What is the probability that the flip was Heads, given that the two drawn marbles have different colors?

Solution

(i) $P(2W) = P(2W | H) P(H) + P(2W | T) P(T) = \frac{1}{2} \left(\frac{3}{52} \left(\frac{2}{51}\right) + \frac{2}{52} \left(\frac{2}{51}\right) \right) = \frac{1}{4}$.

(ii) $P(\text{one R, one W}) = \frac{1}{2} \frac{2\left(\frac{3}{52}\right) \left(\frac{2}{51}\right)}{\left(\frac{3}{52}\right) \left(\frac{2}{51}\right)} \frac{2\left(\frac{2}{52}\right) \left(\frac{3}{51}\right)}{\left(\frac{2}{52}\right) \left(\frac{2}{51}\right)} = \frac{24}{125} + \frac{4}{6} = \frac{3}{5}$, so we get

$P(H | \text{one R, one W}) = \frac{P(H \cap \text{one R, one W})}{P(\text{one R, one W})} = \left(\frac{1}{2} \frac{24}{125}\right) / \frac{4}{9} = \frac{4}{9}$.

5.3 Two teams A and B compete in a “best-of-5” competition. This means
they play each other until one team has won 3 games. Suppose that for any of the games, the probability that A beats B is $\alpha$. What is the probability that A wins the “best-of-5” competition?

**Solution**

Assuming A wins the competition, the last game will be won by A. We now split the problem into cases according to how many games it takes for A to win the competition:

. 3 games all of A wins

. 4 games, implying A wins 2 of the first 3.

. 5 games, implying A wins only 2 of the first 4.

When we sum this up, we get

$$\Pr(\text{AAA}) + \binom{3}{1} \Pr(\text{AABA}) + \binom{4}{2} \Pr(\text{AABBA}) = \alpha^3 + 3\alpha^3(1-\alpha) + 6\alpha^3(1-\alpha)^2).$$

5.4 A fair coin if flipped 3 times. If $(F_1, F_2, F_3)$ denotes a typical flip sequence, let $E_1$ denote the event that at least two of the $F_i$'s are Heads, let $E_2$ denote the event that exactly two of the $F_i$'s are Heads, and let $E_3$ denote the event that all the $F_i$ are the same. Which of the pairs of these three events are independent?

**Solution**

The only pair that is independent is $E_1$ and $E_3$. Indeed, since $\Pr(E_1) = \left(\frac{1}{2}\right)^3 + \binom{3}{2} \left(\frac{1}{2}\right)^3 = \frac{1}{2}$ and $\Pr(E_3) = 2 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{4}$, so $\Pr(E_1 \cap E_3) = \Pr(\text{all Heads}) = \frac{1}{8} = \Pr(\overline{E_1})\Pr(E_3)$, while $\Pr(E_1 \cap E_2) = \Pr(E_2)$ and $\Pr(E_2 \cap E_3) = 0$.

5.5 Two random cards are drawn one at a time without replacement from a deck of 52.

(i) What is the probability that the second card is an Ace?

(ii) What is the probability that the second card is an Ace, given that the first card drawn was a King?

(iii) What is the probability that the second card is an Ace, given that the first card drawn was an Ace?
Solution

(i) $\frac{4}{52}$;
(ii) $\frac{3}{51}$;
(iii) $\frac{2}{51}$

5.6 A biased coin $C$ has $Pr(Heads) = \alpha$ and $Pr(Tails) = 1 - \alpha$. The coin is flipped $n$ times.
What is the expected number of Heads that will occur?
(Optional) What is the expected number of times that the sequence $HT$ will occur? (For example, in the sequence $HHTHHTHTT$, $HT$ occurs 3 times.)

Solution

The expected number of Heads that will occur is $n\alpha$;
Let $F_1, F_2, F_3, \ldots F_n$ denote the outcomes of the $n$ flips and let’s define the random variable $X_i$, where $i = 1, 2, 3, \ldots n - 1$ with $X_i = 1$ if $F_i = H$ and $F_{i+1} = T$. Then $X = X_1 + X_2 + \ldots + X_{n-1}$ will be the number of times the sequence $HT$ occurs, $E(X_i) = \alpha(1 - \alpha)$ and $E(X) = E(X_1) + E(X_2) + \ldots + E(X_{n-1}) = (n - 1)\alpha(1 - \alpha)$. 