Solutions to the Practice Final Exam

1. Solution:
(a) \(a(0) = 1, a(1) = 2, a(2) = 8, a(3) = 31, a(4) = 117\) and \(a(5) = 438\).

(b) Notice that the characteristic polynomial of the recurrence relation is \(x^2 - 4x + 1\), which has roots \(2 \pm \sqrt{3}\), yielding template solution \(a(n) = \alpha (2 + \sqrt{3})^n + \beta (2 - \sqrt{3})^n + \gamma\). From the recurrence and the initial values, we have

\[
\begin{align*}
\gamma &= 4\gamma - 1 \\
\alpha + \beta + \gamma &= 1 \\
\alpha (2 + \sqrt{3}) + \beta (2 - \sqrt{3}) + \gamma &= 2
\end{align*}
\]

which has solution \(\alpha = \frac{1}{4}(3 - \frac{1}{\sqrt{3}}), \beta = \frac{1}{4}(3 + \frac{1}{\sqrt{3}}), \gamma = -\frac{1}{2}\). Thus, we have

\[
a(n) = \frac{1}{4} \left( (3 - \frac{1}{\sqrt{3}})(2 + \sqrt{3})^n + (3 + \frac{1}{\sqrt{3}})(2 - \sqrt{3})^n - 2 \right).
\]

(c) Proof by induction.

Step 1. (Base case) It is easy to verify that \(a(0) = 1\) and \(a(1) = 2\).

Step 2. (Induction) Suppose \(a(n) = \frac{1}{4} \left( (3 - \frac{1}{\sqrt{3}})(2 + \sqrt{3})^n + (3 + \frac{1}{\sqrt{3}})(2 - \sqrt{3})^n - 2 \right)\) for all values up to some \(n \geq 1\). We then compute

\[
a(n + 1) = 4a(n) - a(n - 1) + 1
\]

\[
= \frac{1}{4} \left( 4 \cdot (3 - \frac{1}{\sqrt{3}})(2 + \sqrt{3})^n + 4 \cdot (3 + \frac{1}{\sqrt{3}})(2 - \sqrt{3})^n - 4 \cdot 2 \right)
\]

\[
- \frac{1}{4} \left( (3 - \frac{1}{\sqrt{3}})(2 + \sqrt{3})^{n-1} + (3 + \frac{1}{\sqrt{3}})(2 - \sqrt{3})^{n-1} - 2 \right) + 1
\]

\[
= \frac{1}{4} \left( (3 - \frac{1}{\sqrt{3}})(2 + \sqrt{3})^{n-1}(4 \cdot (2 + \sqrt{3}) - 1) \right)
\]

\[
+ \frac{1}{4} \left( (3 + \frac{1}{\sqrt{3}})(2 - \sqrt{3})^{n-1}(4 \cdot (2 - \sqrt{3}) - 1) \right) - \frac{1}{4}(8 - 2 - 4)
\]

\[
= \frac{1}{4} \left( (3 - \frac{1}{\sqrt{3}})(2 + \sqrt{3})^{n+1} + (3 + \frac{1}{\sqrt{3}})(2 - \sqrt{3})^{n+1} - 2 \right)
\]

and the induction step is complete.

2. Solution:
We guess that the general solution is of the form \(x(n) = \gamma \cdot 3^n + \alpha n + \beta\) for suitable constants \(\alpha, \beta\) and \(\gamma\). Substituting this into the recurrence, we obtain

\[
x(n + 1) = \gamma \cdot 3^{n+1} + \alpha (n + 1) + \beta
\]

\[
= \gamma \cdot 3^{n+1} + \alpha n + \alpha + \beta
\]

\[
= 3 \cdot x(n) + n
\]

\[
= 3 \cdot (\gamma \cdot 3^n + \alpha n + \beta) + n
\]

\[
= \gamma \cdot 3^{n+1} + (3\alpha + 1)n + 3\beta
\]
Thus, we must have

\[
\alpha = 3\alpha + 1, \quad \text{and} \quad \alpha + \beta = 3\beta
\]

which implies \( \alpha = -\frac{1}{2} \) and \( \beta = -\frac{1}{4} \). Since \( x(0) = \gamma \cdot 3^0 - \alpha \cdot 0 - \beta = 1 \) then we find \( \gamma = \frac{5}{4} \) so that \( x(n) = \frac{1}{4}(5 \cdot 3^n - 2n - 1) \). (Check small values for reassurance, or prove it by induction).

3. Solution:

With probability \( \frac{1}{2} \) (coin comes up Heads), we have 5 Red, 5 White, and 5 Blue.

With probability \( \frac{1}{2} \) (coin comes up Tails), we have 3 Red, 3 White, and 5 Blue.

(a) \[
\frac{1}{2} \frac{\binom{5}{1} \binom{5}{1} \binom{5}{1}}{\binom{15}{3}} + \frac{1}{2} \frac{\binom{3}{1} \binom{3}{1} \binom{5}{1}}{\binom{11}{3}}
\]

(b) \[
\frac{\Pr(\text{three marbles have the same color}|\text{Heads})}{\Pr(\text{three marbles have the same color, Heads})} = \frac{\binom{5}{3} + \binom{3}{3} + \binom{5}{3}}{\binom{15}{3}}
\]

(c) \[
\frac{\Pr(\text{Heads}|\text{three marbles have the same color})}{\Pr(\text{three marbles have the same color, Heads})} = \frac{(\binom{5}{3} + \binom{3}{3})/(\binom{15}{3})}{(\binom{5}{3} + \binom{3}{3} + \binom{5}{3})/(\binom{15}{3}) + (\binom{3}{3} + \binom{5}{3} + \binom{3}{3})/(\binom{11}{3})}.
\]

4. Solution

Let the expected number of flips for this to first happen be \( x \). Consider the following decision tree.

```
Start flipping coins
  ↓
H → T_1
  ↓
H → T_2
```

Notice that the expected number of flips for 2 consecutive H’s to first happen from either of the positions \( T_1 \) or \( T_2 \) in the decision tree is also \( x \). Since the probability of \( T_1 \) occurring is \( \frac{1}{2} \), and the probability of \( T_2 \) occurring is \( \frac{1}{4} \), we have the following equation.

\[
x = \Pr(T_1) \cdot (1 + x) + \Pr(HT_2) \cdot (2 + x) + \Pr(HH) \cdot 2
\]

\[
x = \frac{1}{2} \cdot (1 + x) + \frac{1}{4} \cdot (2 + x) + \frac{1}{4} \cdot 2.
\]

Solving for \( x \) gives \( x = 6 \).
5. Solution
We will use the abbreviation that $a^r$ denotes the quantity $\frac{a!}{(a-r)!}$.

Case 1. The number of passwords without digits is $26^5$.

Case 2. The number of passwords with 1 digit is $\binom{5}{1} \cdot 10 \cdot 26^4$.

Case 3. The number of passwords with 2 digits is $\binom{5}{2} \cdot 10 \cdot 9 \cdot 26^3$.

Thus, the total number of passwords is the sum of these three expressions.

6. Solution
Proof by induction.

Step 1. (Base case) First note that $\sum_{k=1}^{1}(2k - 1)^2 = 1 = \binom{2 \times 1 + 1}{3}$

Step 2. (Induction) Suppose $\sum_{k=1}^{n}(2k - 1)^2 = \binom{2n+1}{3}$. Then

$$\sum_{k=1}^{n+1}(2k - 1)^2 = (2(n + 1) - 1)^2 + \sum_{k=1}^{n}(2k - 1)^2$$

$$= (2n + 1)^2 + \binom{2n + 1}{3}$$

$$= (2n + 1)^2 + \frac{(2n + 1)(2n)(2n - 1)}{3!}$$

$$= (2n + 1)\frac{12n + 6 + (2n)(2n - 1)}{6}$$

$$= (2n + 1)\frac{(2n + 2)(2n + 3)}{6}$$

$$= \frac{(2(n + 1) + 1)}{3}$$

and the induction step is complete.

7. Solution
There are two ways to think about this problem. The first is a straightforward approach. Since the probability of picking $i$ good bulbs is $\binom{\binom{5}{i}}{(\binom{5}{4})}$, we have

$$\sum_{i=0}^{4} \binom{\binom{5}{i}}{(\binom{5}{4})} \cdot i = 2.$$ 

where you have to do a bit of simplification in the sum to get the desired answer.

Another (simpler) approach is to use linearity of expectation by writing the random variable $X$ which counts the number of good bulbs selected as the sum $X = \sum_{i=1}^{4} X_i$, where $X_i$ is 1 if the $i$th bulb selected is good, and 0 if not. Since $E(X_i) = 1/2$, then by linearity of expectation, $E(X) = 4 \cdot 1/2 = 2$.

8. Solution
First, let’s count the number ways of distributing the pens. Giving the CSE students each 2 pens to begin with and 1 each to the three ECE students leaves just $15 - 13 = 2$ pens left to distribute to 10 students, and the number of ways to do this is $\binom{2+10-1}{10} = \binom{11}{10}$.

Now to distribute the marbles. First, we give each of the ECE students one marble, leaving 22 marbles left to distribute.

Case 1. No Bio student gets a marble: Then the number of ways is $\binom{22+7}{7}$;
case 2. Only the first Bio student gets a marble: Then the number of ways is \( \binom{21}{7} \);

case 3. Only the second Bio student gets a marble: Then the number of ways is \( \binom{21}{7} \);

case 4. Both Bio students get a marble: Then the number of ways is \( \binom{20}{7} \);

Thus, the total number of ways of distributing pens and marbles is
\[
\left( \frac{11}{2} \right) \cdot \left( \frac{29}{7} \right) + 2 \cdot \left( \frac{28}{7} \right) + \left( \frac{27}{7} \right).
\]

9. Solution
Count the complement. We can assume that \( B \geq 2 \) since otherwise such a seating is impossible. The total number of ways for \( B \) boys and 3 girls to sit in a row is \((B + 3)\).

case 1. The total number of ways that three girls sit together is \((B + 1) \cdot 3! \cdot B!\);

case 2. The number of way that two girls in the two leftmost seats, and the third girl sits with at least one boy separating them is \(B \cdot 3! \cdot B!\).

case 3. The number of way that two girls sit together in the two rightmost seats, and the third girl sits with at least one boy separating them is also \(B \cdot 3! \cdot B!\).

Thus, the total number of ways is:
\[
(B + 3)! - (B + 1) \cdot 3! \cdot B! - 2B \cdot 3! \cdot B!.
\]

10. Solution
(a) \(3^6\).
(b) \(S(6,3) \cdot 3! = 90 \cdot 6 = 540\), where \(S(6,3)\) is the Stirling number that counts the number of ways of partitioning a 6-set into 3 nonempty subsets.
(c) \(\frac{6!}{2!2!2!}\).

11. Solution
\[
\frac{\Pr(\text{“actually good”} \mid \text{tests good})}{\Pr(\text{tests good} \& \text{“actually good”})} = \frac{\Pr(\text{“tests good”})}{\frac{499}{500} \times \frac{9}{10} + \frac{1}{500} \times \frac{1}{100}}.
\]

12. Solution
By using the greedy algorithm on the graph (and adding some small numbers correctly), you should get a total of \(7 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 + 6 = 68\).

13. Solution
(d) \(f \circ g \circ h\)

Solution
Decomposing \(X = \sum_{i=1}^{5} X_i\) in the usual way (so that \(X_i = 1\) if and only if the \(i^{th}\) flip is Heads), we find that \(E(X_i) = 2/3\) and \(\text{Var}(X_i) = 2/9\). Thus, \(E(X) = 10/3\) and (since the \(X_i\) are independent), \(\text{Var}(X) = 10/9\).