1 [20 points]
How many different arrangements are there of all the letters in the word BOOKKEEPER?

In this homework, we will consider “ordinary” decks of playing cards which have 52 cards, with 13 of each of the four suits (Hearts, Spades, Diamonds and Clubs), with each suit having the 13 ranks (Ace, 2, 3, ..., 9, 10, Jack, Queen, King).

(a) $10!$

(b) $6! \times 3! \times 2! \times 2!$

(c) $\frac{10!}{3!2!2!}$

(d) $\frac{10!}{3!2!2!2!2!}$

(e) none of the above

The answer is (c).

There are 10 letters, so there are $10!$ ways to arrange 10 unique letters. Now since there are 2 O’s, 2 K’s, and 3 E’s, we must divide by $3! \times 2! \times 2!$ to account for the repetition of these letters.
An urn contains 2 red balls, 3 white balls, and 4 blue balls. You reach into the urn and randomly remove three balls from the urn (without replacement). What is the probability that at least two of the balls removed have the same color?

(a) \( \frac{4}{7} \)
(b) \( \frac{5}{7} \)
(c) \( \frac{3}{7} \)
(d) \( \frac{1}{2} \)
(e) none of the above

The answer is (b).

Let \( A = \) Probability at least two balls are removed have the same color. Then, \( P(A^c) \) is the probability that all the ball have different colors.

\[
P(A^c) = \frac{\binom{2}{1}\binom{4}{1}\binom{3}{1}}{\binom{9}{3}}
\]

\[
P(A) = 1 - P(A^c) = 1 - \frac{2}{7} = \frac{5}{7}
\]

\[\square\]
3 [20 points]
5 cards are selected without replacement at random from an ordinary deck of 52 cards. What is the probability that at most one of the card is a Heart?

(a) \( \frac{39^5 + 13 \cdot 39^5}{52^5} \)
(b) \( \frac{\binom{39}{5} + 13 \cdot \binom{39}{4}}{\binom{52}{5}} \)
(c) \( \frac{\binom{39}{5}}{\binom{52}{5}} \)
(d) \( \frac{39^5 + 13 \cdot 39^4}{52^5} \)
(e) none of the above

The answer is (b).

The probability at most one card is a heart is equivalent to the probability 1 card is a heart or 0 cards are a heart.

\[
P(\text{no heart}) = \frac{\binom{39}{5}}{\binom{52}{5}}
\]

\[
P(\text{1 heart}) = \frac{\binom{52}{5}}{\binom{52}{5}}
\]

\[
P(\text{no heart}) + P(\text{1 heart}) = \frac{\binom{39}{5} + 13 \cdot \binom{39}{4}}{\binom{52}{5}}
\]

4 [20 points]
(a) How many 5-card hands (from an ordinary deck) have at least one card of each suit?
(b) How many 6-card hands have at least one card of each suit?

The answer is (a).

Use stars and bars. Consider distributing jellybeans separate from cookies.
Once every boy gets at least 1 cookie, and every girl at least 2, there are 9 cookies left (9 stars).
Give them to 4 boys (3 bars), thats \( \binom{12}{3} \) ways.
Once every girl gets at least 3 jellybeans, there are 5 left (5 stars).
Give them to 4 boys and 5 girls (8 bars), that \( \binom{13}{8} \) ways.
Combine the two results to get \( \binom{12}{3} \binom{13}{8} \).

5 [20 points]
Seven light bulbs are selected at random from a set of 10 light bulbs of which 4 are defective. What is the probability that at least two of the selected bulbs are defective?

Let \( A = \) Probability at least two selected bulbs are defective. Then, \( P(A^c) \) is the probability that zero or one bulbs are defective.

\[
P(A^c) = 0 + \binom{4}{1} \binom{6}{6} \binom{10}{7} = \frac{4}{120} = \frac{1}{30}
\]

\[
P(A) = 1 - P(A^c) = 1 - \frac{1}{30} = \frac{29}{30}
\]

6 [20 points]
Allen, Bob, Charlie are playing darts. Allen is pretty good and can hit the bullseye 80\% of the time. However, Bob and Charlie are not as good and they hit the bullseye 50\% of the time. They each throw one dart at the target.

(a) What is the probability that all three darts hit the bullseye?
(b) What is the probability that at least one of the darts hits the bullseye?
(c) What is the probability that at most one of the darts hits the bullseye?

Let \( A = \) Allen hits the target. Let \( B = \) Bob hits the target. Let \( C = \) Charlie hits target.
(a) \( P(A \cap B \cap C) = 0.8 \times 0.5 \times 0.5 = 0.20 \)
(b) \( P(A^c \cap B^c \cap C^c) = 1 - (0.2 \times 0.5 \times 0.5) = 0.95 \)
(c) Probability 0 hits: \( P(A^c \cap B^c \cap C^c) = 0.05 \)
    Probability 1 hit: \( P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) = 0.05 + 0.2 + 0.05 + 0.05 = 0.35 \)

7 [20 points]
In how many ways can 3 boys and 2 girls sit in a row if:

(a) Every boy is seated next to at least one girl.
(b) Every girl is seated next to at least one boy.

(a) \( 3! \times 2! \times 2! \). 3 arrangements, BGBBG, BGBGB, GBBGB. 3! ways to arrange the boys, 2! ways to arrange the girls.

(b) \( 8 \times 3! \times 2! \). 8 arrangements. BBGBG, BBGGB, BGBBG, BGBGB, BGGBB, GBGGG, GBGBB, GBBGB. 3! ways to arrange the boys, 2! ways to arrange the girls.

8 [20 points]
Consider the three permutations given in cycle form:
\( f = (124)(35), g = (13)(45)(2), h = (1)(4253) \) Which of the compositions gives the permutation \( (1)(2)(34)(5) \)?

(a) \( f \circ h \circ g \circ f \circ g \)
(b) \( g \circ f \circ h \circ g \circ f \)
(c) \( h \circ f \circ g \circ f \circ h \)
(d) \( f \circ g \circ h \circ g \circ f \)
(e) none of the above.
The answer is (a).
Use process of elimination to answer this question. Start with number 1, and make sure that it maps back to itself. By that point, you should be able to cross off choices (a) and (c). Now you are left with (b) and (d). Oops (b) doesn’t work! Now check that all numbers are valid for (d), which they are.