6.1 An urn contains 3 Red and 4 Blue marbles. A random marble $M_1$ is drawn. If $M_1$ is Red then 2 more Blue marbles are placed in the urn (but $M_1$ is not put back into the urn). If $M_1$ is Blue, then it is placed back into the urn along with 3 other Red marbles. Now, another marble $M_2$ is randomly drawn from the urn.

(a) What is the probability that $M_2$ is Blue?
(b) What is the probability that $M_2$ is Red given that $M_1$ is Blue?
(c) What is the probability that $M_1$ is Blue given that $M_2$ is Red?

\[
Pr(M_2 \text{ is Blue}) = \frac{3}{7} \cdot \frac{3}{4} + \frac{4}{7} \cdot \frac{2}{5} = \frac{9}{28} + \frac{8}{35} = \frac{11}{20}
\]
(b) \[
Pr(M_2 \text{ is Red} \mid M_1 \text{ is Blue}) = \frac{Pr(M_1 \text{ is Blue} \cap M_2 \text{ is Red})}{Pr(M_1 \text{ is Blue})} = \frac{4}{7} \cdot \frac{3}{5} = \frac{3}{5}
\]

(c) \[
Pr(M_1 \text{ is Blue} \mid M_2 \text{ is Red}) = \frac{Pr(M_2 \text{ is Red} \mid M_1 \text{ is Blue}) \cdot Pr(M_1 \text{ is Blue})}{Pr(M_2 \text{ is Red})} = \frac{3}{5} \cdot \frac{4}{7} = \frac{16}{21}
\]

6.2 We imagine that UCSD and Stanford are playing for the National Ultimate Frisbee Championship. The rules are that a maximum of 5 games will be played, and the first team winning 3 games will be declared the champion. Suppose that the probability that UCSD beats Stanford in any individual game is 0.6. What is the probability that UCSD will become the champion?

There are three disjoint possible scenarios:

- UCSD wins the first 3 games and thus the Championship.
- UCSD wins exactly 2 of the first 3 games and then has its 3rd win in the fourth game.
- UCSD only wins 2 of the first 4 games, but wins the fifth and final one.

The combined probability for any of these three scenarios occurring is
\[
\left( \frac{3}{5} \right)^3 + \left( \frac{3}{5} \right)^2 \cdot \frac{2}{5} \cdot \frac{3}{5} + \left( \frac{4}{2} \right) \left( \frac{3}{5} \right)^2 \cdot \left( \frac{2}{5} \right) \cdot \frac{3}{5} = \frac{27}{25} \cdot \frac{79}{55}
\]
We are given an urn that has one red ball and two white balls in it. A fair die is thrown. If the number shown is 1 or 2 then one red ball is added to the urn. Otherwise three red balls are added to the urn. A ball is then randomly drawn from the urn.

(a) Given that a red ball was drawn, what is the probability that a 1 or 2 appeared when the die was thrown?

(b) Given that the final composition of the urn contained more than one red ball, what is the probability that a 1 or 2 appeared when the die was thrown?

(a) Note first that \( Pr(\text{Red}) = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{2}{3} = \frac{11}{18} \). We then have

\[
Pr(1 \text{ or } 2 | \text{Red}) = \frac{Pr(\text{Red} | 1 \text{ or } 2) \cdot Pr(1 \text{ or } 2)}{Pr(\text{Red})}
\]

\[
= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{11}{18}} = \frac{3}{11}
\]

(b) Let’s name the event corresponding to a final composition of more than one red ball left in the end, \( A \). Then \( A^C \) corresponds to the single leaf to the left in the decision tree (or the probability of having only one red ball left in the end) and the probability of this occurring is \( Pr(A^C) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \), so \( Pr(A) = 1 - \frac{1}{6} = \frac{5}{6} \). Then

\[
Pr(1 \text{ or } 2 | A) = \frac{Pr(A | 1 \text{ or } 2) \cdot Pr(1 \text{ or } 2)}{Pr(A)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{5}{6}} = \frac{1}{5}.
\]
The weather in Saskatoon has the following properties. Every day is either Fair or Rainy. If some day is Fair, then the probability that the following day is also Fair is 0.6, and the probability that the following day is Rainy is 0.4. On the other hand, if some day is Rainy, then the probability that the following day is Rainy is 0.8, and the probability that it is Fair is 0.2.

Suppose some particular Monday is Fair.

(a) What is the probability that the next Thursday is Fair?

(b) What is \( Pr(\text{Tuesday is Fair} \mid \text{Thursday is Rainy}) \)?

(a) Starting with a Fair Monday and ending on a Fair Thursday, the following configurations are possible: Fair Fair Fair Fair, Fair Fair Rainy Fair, Fair Rainy Fair Fair and Fair Rainy Rainy Fair.

The combined probability will be

\[
\left(\frac{3}{5}\right)^3 + \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{1}{5} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} = \frac{27 + 6 + 6 + 8}{125} = \frac{47}{125}.
\]

(b) \[
Pr(\text{Tue Fair} \mid \text{Thu Rainy}) = \frac{Pr(\text{Thu Rainy} \mid \text{Tue Fair}) \cdot Pr(\text{Tue Fair})}{Pr(\text{Thu Rainy})}.
\]

We can compute \( Pr(\text{Thu Rainy} \mid \text{Tue Fair}) = \frac{3}{3} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{4}{5} = \frac{14}{25} \), and we know that \( Pr(\text{Tue Fair}) = \frac{3}{5} \), because Monday is Fair. Furthermore, using (a) we get

\[
Pr(\text{Thu Rainy}) = 1 - Pr(\text{Thu Fair}) = 1 - \frac{47}{125} = \frac{78}{125},
\]

so we finally have

\[
Pr(\text{Tue Fair} \mid \text{Thu Rainy}) = \frac{\frac{14}{25} \cdot \frac{3}{5}}{\frac{78}{125}} = \frac{42}{78} = \frac{7}{13}.
\]
An urn initially contains 3 red iPods and 2 blue iPods.

(a) You draw out at random one of the iPods. What is the probability that you get a red iPod?

(b) Before you draw, someone secretly removes 2 of the iPods (randomly) from the urn. Now you draw out one at random one of the remaining iPods. What is the probability now that you get a red iPod?

(a) \( P(\text{red iPod}) = \frac{3}{5} \)

(b) \[
P(\text{red iPod}) = \frac{\binom{3}{2} \frac{1}{3} + \binom{2}{1} \binom{3}{1} \frac{2}{3} + \binom{2}{2} \frac{1}{3} \frac{1}{5}}{\binom{5}{2}}
\]

It makes sense that the probability would be the same for part (a) and part (b) because the two secretly removed iPods are taken out randomly.

\[ \boxed{\frac{3}{5}} \]
The Acme Grommet Company manufactures grommets (whatever they are!). Unfortunately, their quality control processes aren't very good, so it turns out that in any batch of grommets produced, the fraction $\alpha$ are defective (= bad), while the fraction $1 - \alpha$ are good. Fortunately, they have a (not very good) test $T$ which behaves as follows. If $T$ is applied to a bad grommet $G$ then it says that $G$ is bad with probability $\beta$, and it says that $G$ is good with probability $1 - \beta$. On the other hand, if $T$ is applied to a good grommet $G$, then it says that $G$ is good with probability $\gamma$, and it says that $G$ is bad with probability $1 - \gamma$. A random grommet $G$ is selected and the test $T$ is applied.

(a) What is $\Pr(G$ is good$)$?

(b) What is $\Pr(T$ says $G$ is good $| G$ is good$)$?

(c) What is $\Pr(G$ is good $| T$ says $G$ is bad$)$?

(a) $1 - \alpha$;

(b) $\gamma$;

(c) 

$$
\Pr(G$ is good $| T$ says $G$ is bad$) = \frac{\Pr(T$ says $G$ is bad $| G$ is good$) \cdot \Pr(G$ is good$)}{\Pr(T$ says $G$ is bad$)}.$$

Now, $\Pr(T$ says $G$ is bad$) = \beta \cdot \alpha + (1 - \gamma)(1 - \alpha)$, thus

$$
\Pr(G$ is good $| T$ says $G$ is bad$) = \frac{(1 - \gamma)(1 - \alpha)}{\beta \cdot \alpha + (1 - \gamma)(1 - \alpha)}.
$$