5.1 A bin contains 4 Red balls, 5 White balls, and 6 blue balls. A random subset S of 4 balls is removed (without replacement). Consider the following 3 events:

(1) $E_1$: S has exactly 2 Red balls.
(2) $E_2$: S has balls of all three colors.
(3) $E_3$: S has at least 2 Blue balls.

Which of the pairs of events $E_1$, $E_2$ and $E_3$ (if any) are independent? (Justify your answer.)

None of the pairs are independent. For example, take the pair $E_1$ and $E_2$. We have to compute $Pr(E_1)$, $P(E_2)$ and $Pr(E_1 \cap E_2)$. Since the sample space has size $\binom{15}{4}$ and all these choices are equally likely, then:

$Pr(E_1) = \frac{\binom{4}{2}\binom{11}{2}}{\binom{15}{4}}, Pr(E_2) = \frac{\binom{4}{2}\binom{5}{5}\binom{6}{4} + \binom{4}{1}\binom{5}{5}\binom{6}{5} + \binom{4}{1}\binom{5}{1}\binom{6}{4}}{\binom{15}{4}}$ and $Pr(E_1 \cap E_2) = \frac{\binom{4}{2}\binom{5}{5}\binom{6}{4}}{\binom{15}{4}}$.

Now we have to check if $Pr(E_1 \cap E_2) = Pr(E_1)Pr(E_2)$. Since this doesn’t hold, then the events $E_1$ and $E_2$ are not independent. Similar calculations for the other two pairs show that they are also not independent.

5.2 3 girls and 3 boys are seated in a row. How many ways can this be done if:

(a) Each girl sits next to at least one boy.
(b) Each girl sits next to at most one boy.
(c) Exactly one girl is sitting next to no boys.
(a) Since the tree is symmetric, we will do the side starting with a girl, and multiply the valid number by 2.

So the total number of templates are $2 \times 5 = 10$. Therefore the final answer is $10 \times 3! \times 3!

(b) Once again, the tree is symmetric, so we will do the side starting with a girl, and multiply the valid number by 2.
So the total number of templates are $2 \times 5 = 10$.
Therefore the final answer is $10 \times 3! \times 3!

(c) Once again, the tree is symmetric, so we will do the side starting with a girl, and multiply the valid number by 2.
So the total number of templates are $2 \times 4 = 8$.
Therefore the final answer is $8 \times 3! \times 3!$

5.3

How many permutations of $\pi$ of $[5] = \{1,2,3,4,5\}$ are there such that $|\pi(k) - \pi(k + 1)| \leq 2$?

We solve this problem using a decision tree. We will let each level be a specific value of $k$, and consider that the resulting permutations are written in two line notation.
Since the permutations are symmetric, then the permutations starting at 1 and 2 have the same form as the permutations starting at 4 and 5. Therefore we have $2^*(9) + 2 = 20$ permutations total.

**Note:** If we ask the same question for permutations of [6], then the answer is 34.

### 5.4

We are given an urn that has two red ball and three white balls in it. Three balls are drawn without replacement.

(a) What is the expected number of red balls that are drawn?

(b) Unknown to you, someone removes a random ball from the urn before you draw your three. Now when you draw three, what is the expected number of red balls drawn?
(a) Our sample space consists of all $5 \times 4 \times 3 = 60$ 3-tuples $(B_1, B_2, B_3)$ of distinct balls. Let $X$ denote the number of red balls drawn. Use the usual trick of decomposing $X = X_1 + X_2 + X_3$ where $X_i = 1$ if $B_i$ is red, and 0 if $B_i$ is white. It is easy to see that the expected value $E(X_i) = \frac{2}{5}$, so by linearity of expectation, $E(X) = \sum_{i=1}^{3} E(X_i) = \frac{6}{5}$.

(b) Now we can let our sample space be the set of all 4-tuples $(R, B_1, B_2, B_3)$ of distinct balls, where $R$ denotes the unknown random ball drawn before we draw our three. As before, if we let $X$ denote the number of red ball drawn among the $B_i$ and we decompose $X = X_1 + X_2 + X_3$ as before, then we still get $E(X_i) = \frac{2}{5}$ and $E(X) = \frac{6}{5}$.

5.5

Two teams A and B compete in a tournament consisting of at most 4 games. The probability that A wins any particular game is $\frac{2}{3}$, and the probability that B wins is $\frac{1}{3}$ (there can be no ties). If either team wins 2 consecutive games, that team is declared the winner. If at the end of 4 games there is no winner, the tournament is declared a draw. What is the expected number of games in the tournament? What is the variance for the number of games played?

\[
E(X) = P(X) \cdot X \\
= \left( \frac{2}{3} \right)^2 + \left( \frac{1}{3} \right)^2 \cdot 2 + \left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 + \left( \frac{2}{3} \cdot \frac{1}{3} \right)^2 \cdot 3 \\
+ \left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 + \left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 + \left( \frac{2}{3} \cdot \frac{1}{3} \right)^2 \cdot 4 \\
= \frac{8}{3}
\]

\[
E(X^2) = \left( \frac{2}{3} \right)^2 + \left( \frac{1}{3} \right)^2 \cdot 2^2 + \left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 + \left( \frac{2}{3} \cdot \frac{1}{3} \right)^2 \cdot 3^2 \\
+ \left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 + \left( \frac{1}{3} \cdot \frac{2}{3} \right)^2 + \left( \frac{2}{3} \cdot \frac{1}{3} \right)^2 \cdot 4^2 \\
= \frac{70}{9}
\]
\[ Var(X) = E(X^2) - [E(X)]^2 \]
\[ = \frac{70}{9} - \left(\frac{8}{3}\right)^2 \]
\[ = \frac{2}{3} \]

### 5.6
A roulette wheel has numbers 1 through 36, and in addition, the two numbers 0 and 00. We assume that all 38 numbers are equally likely to be where the ball finally ends up. Of course, half of the numbers 1 through 36 are Odd, and half of them are Even. If you bet $k$ on Odd and win, the payoff is $2k$ (and the same applies to Even). Suppose you use the following strategy: You first bet $10$ on Odd. If you win, you collect $20$ and quit. On the other hand, if you lose then you bet $20$ on Odd. Again, if you win, you collect $40$ and quit. If not, then you bet one more time on Odd, this time betting $40$. After this round, you quit, no matter whether you win or lose. What is expected amount you win playing this strategy.

There are four possible outcomes: 3 of the outcomes will cause you to gain $10, 1 of the outcomes will cause you to lose $70. So using this knowledge and plugging in the probability values we get:

\[ E(X) = \left(\frac{18}{38}\right) \times 10 + \left(\frac{20}{38} \times \frac{18}{38}\right) \times 10 + \left(\frac{20}{38} \times \frac{18}{38} \times \frac{20}{38}\right) \times 10 \]
\[ - \left(\frac{20}{38}\right)^3 \times 70 \]
\[ = 8.542 - 10.205 \]
\[ = -1.663 \]