MAX 3SAT

3-SAT Problem: Given a boolean formula F consisting of:
- n variables $x_1, x_2, ..., x_n$
- m clauses of size 3 of the form $x_i \lor x_j \lor x_k$ or $\neg(x_i) \lor x_j \lor x_k$
Is there an assignment of true/false values to variables s.t. all clauses are true?

Example:
3 variables: $x_1, x_2, x_3$
Clauses:
$x_1 \lor x_2 \lor x_3$, $\neg(x_1) \lor x_2 \lor x_3$, $x_1 \lor \neg(x_2) \lor x_3$, $x_1 \lor x_2 \lor \neg(x_3)$, $\neg(x_1) \lor \neg(x_2) \lor x_3$, $\neg(x_1) \lor x_2 \lor \neg(x_3)$, $x_1 \lor \neg(x_2) \lor \neg(x_3)$, $\neg(x_1) \lor \neg(x_2) \lor \neg(x_3)$
Unsatisfiable!

MAX-3SAT Problem: Given a boolean formula F consisting of:
- n variables $x_1, x_2, ..., x_n$
- m clauses of size 3 of the form $x_i \lor x_j \lor x_k$ or $\neg(x_i) \lor x_j \lor x_k$
Find an assignment of true/false values to variables to satisfy the most clauses

Example:
Any assignment satisfies 7 out of 8 clauses
**MAX 3SAT**

**MAX-3SAT Problem:** Given a boolean formula $F$ consisting of:

- $n$ variables $x_1, x_2, \ldots, x_n$
- $m$ clauses of size 3 of the form $x_i \lor x_j \lor x_k$ or not($x_i$) $\lor x_j \lor x_k$

Find an assignment of true/false values to variables to satisfy the most clauses

**A Randomized MAX-3SAT Algorithm:**

Set each variable to 0/1 independently with probability 1/2 each

Define: $Z_i = 1$, if clause $i$ is satisfied by the assignment  
$Z_i = 0$, otherwise

$\Pr[Z_i = 0] = (1/2) \cdot (1/2) \cdot (1/2) = 1/8$

$E[Z_i] = 7/8$

Let $Z = Z_1 + Z_2 + \ldots + Z_m = \#satisfied \ clauses$

$E[Z] = E[Z_1 + Z_2 + \ldots + Z_m] = E[Z_1] + E[Z_2] + \ldots + E[Z_m] = 7m/8 = E[#satisfied \ clauses]$

How to get a solution with $\geq 7m/8$ satisfied clauses?

**Fact:** $P = \Pr[Solution \ has \ \geq 7m/8 \ satisfied \ clauses] \geq 1/8m$

**Proof:** Let $p_j = \Pr[Solution \ has \ j \ satisfied \ clauses]$, $k = \text{largest integer} < 7m/8$

$$\frac{7m}{8} = E[Z] = \sum_{j=0}^{k} j p_j + \sum_{j=k+1}^{m} j p_j \leq k + mP \quad \rightarrow \quad P \geq \frac{\frac{7m}{8} - k}{m} \geq \frac{1}{8m}$$

As $m, k$ are integers, $7m/8 - k \geq 1/8$
**MAX-3SAT Problem:** Given a boolean formula $F$ consisting of:
- $n$ variables $x_1, x_2, \ldots, x_n$
- $m$ clauses of size 3 of the form $x_i \lor x_j \lor x_k$ or $\neg(x_i) \lor x_j \lor x_k$
Find an assignment of true/false values to variables to satisfy the most clauses

**A Randomized MAX-3SAT Algorithm:**
Set each variable to 0/1 independently with probability $1/2$ each

Define: $Z_i = 1$, if clause $i$ is satisfied by the assignment
$Z_i = 0$, otherwise
$\Pr[Z_i = 0] = (1/2) \cdot (1/2) \cdot (1/2) = 1/8$
$E[Z_i] = 7/8$

Let $Z = Z_1 + Z_2 + \ldots + Z_m = \#satisfied\ clauses$
$E[Z] = E[Z_1 + Z_2 + \ldots + Z_m] = E[Z_1] + E[Z_2] + \ldots + E[Z_m] = 7m/8 = E[\#satisfied\ clauses]$

How to get a solution with $\geq 7m/8$ satisfied clauses?

**Fact:** $\Pr[Solution\ has\ \geq 7m/8\ satisfied\ clauses] \geq 1/8m$

**Solution:** Run algorithm $8m \log(1/t)$ times independently. W.p. $1 - t$, there will be a solution with at least $7m/8$ satisfied clauses.
Variance

Given a random variable $X$, its variance $\text{Var}[X]$ is defined as:  
$$\text{Var}(X) = E[(X - E[X])^2]$$

Variance of a random variable measures its “spread”

High Variance Distribution

Low Variance Distribution
Variance

Given a random variable $X$, its variance $\text{Var}[X]$ is defined as:

$$\text{Var}(X) = E[(X - E[X])^2]$$

**Examples:**

1. Let $X = 1$ if a fair coin toss comes up heads, 0 otherwise. What is $\text{Var}(X)$?
Given a random variable $X$, its variance $\text{Var}[X]$ is defined as:

$$\text{Var}(X) = E[(X - E[X])^2]$$

**Examples:**

1. Let $X = 1$ if a fair coin toss comes up heads, 0 otherwise. What is $\text{Var}(X)$?
2. Let $X =$ outcome of a fair dice throw. What is $\text{Var}(X)$?
**Variance**

Given a random variable $X$, its variance $\text{Var}[X]$ is defined as:

$$\text{Var}(X) = E[(X - E[X])^2]$$

**Properties of Variance:**

1. $\text{Var}(X) = E[X^2] - (E[X])^2$
Variance

Given a random variable $X$, its variance $\text{Var}[X]$ is defined as:  
\[ \text{Var}(X) = E[(X - E[X])^2] \]

**Properties of Variance:**

1. $\text{Var}(X) = E[X^2] - (E[X])^2$

2. If $X$ and $Y$ are independent random variables, then, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
Variance

Given a random variable $X$, its variance $\text{Var}[X]$ is defined as: $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$

Properties of Variance:

1. $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
2. If $X$ and $Y$ are independent random variables, then, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
3. For any constants $a$ and $b$, $\text{Var}(aX + b) = a^2 \text{Var}(X)$
Variance

Given a random variable $X$, its variance $\text{Var}[X]$ is defined as:

$$\text{Var}(X) = E[(X - E[X])^2]$$

If $X$ and $Y$ are independent random variables, then,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

What if $X$ and $Y$ are not independent?

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = E( (X - E[X]) (Y - E[Y]) )$$

$\text{Cov}(X,Y)$ measures how closely $X,Y$ are “correlated” (in a loose sense)

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X,Y) \quad \text{[for general r.v. } X \text{ and } Y \text{]}$$

What is $\text{Cov}(X,Y)$ if $X$ and $Y$ are independent?
Concentration Inequalities

1. Measures how often a random variable is close to its expectation

2. This class: three inequalities, progressively tighter, but have progressively more restrictions
Inequality 1: Markov’s Inequality

If \(X\) is a random variable which takes non-negative values, and \(a > 0\), then

\[
\Pr[X \geq a] \leq \frac{E[X]}{a}
\]

**Example:** \(n\) tosses of an unbiased coin. \(X = \#\text{heads}\)

\(E[X] = n/2\). Let \(a = 3n/4\). By Markov’s Inequality, \(\Pr(X \geq a) \leq 2/3\). But what is it really?

\[
\Pr[X \geq \frac{3n}{4}] \leq \left(\left(\frac{n}{3n/4}\right) + \left(\frac{n}{3n/4 + 1}\right) + \ldots + \left(\frac{n}{n}\right)\right) \cdot 2^{-n} \leq n \cdot 2^{-n} \cdot \left(\frac{n}{n/4}\right)
\]

\[
\leq n \cdot 2^{-n} \cdot (4e)^{n/4} \leq n(e/4)^{n/4} < (e/3)^{n/4} \quad \text{for large } n
\]

**Fact:** If \(n \geq k\)

\[
\left(\frac{n}{k}\right)^k \leq \left(\frac{e}{k}\right)^k \leq \left(\frac{ne}{k}\right)^k
\]

**Summary:** Markov’s inequality can be weak, but it only requires \(E[X]\) to be finite!