CSE 202: Design and Analysis of Algorithms

Lecture 14

Instructor: Kamalika Chaudhuri
Algorithm Design Paradigms

• **Exhaustive Search**

• **Greedy Algorithms:** Build a solution incrementally piece by piece

• **Divide and Conquer:** Divide into parts, solve each part, combine results

• **Dynamic Programming:** Divide into subtasks, perform subtask by size. Combine smaller subtasks to larger ones

• **Hill-climbing:** Start with a solution, improve it

• **Randomized Algorithms:** Algorithm can make random choices
Randomized Algorithms

- Algorithm can make random decisions
- Why randomized algorithms? Simple and efficient
- Examples: Symmetry-breaking, graph algorithms, quicksort, hashing, load balancing, cryptography, etc
Contestion Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol.

Restriction: Processors can’t communicate.
Simultaneous access blocks the resource.

Processors

Object

1
2
3
4
5

Need symmetry-breaking!
Given $n$ processors and a resource which they all wish to access, design a symmetry breaking access protocol.

Restriction: Processors can’t communicate.
Simultaneous access blocks the resource.

**Protocol:** At time $t$, each processor accesses resource w.p. $p = 1/n$.
Contention Resolution

Given $n$ processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processes can't communicate
   Simultaneous access blocks the resource

**Protocol:** At time $t$, each processor accesses resource w.p. $p = 1/n$

**Goals:**
- Low contention
- Low waiting time for all processors
Contestation Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol.

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. p = 1/n

**Goals:**
- Low contention
- Low waiting time for all processors

S(i, t) = Event that processor i succeeds at time t = At time t, only processor i accesses
Contention Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
  Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. \( p = \frac{1}{n} \)

**Goals:**
- Low contention
- Low waiting time for all processors

\( S(i, t) = \) Event that processor i succeeds at time t = At time t, only processor i accesses

\[
\Pr[S(i, t)] = p \times (1 - p)^{n-1} = \left( \frac{1}{n} \right) \times \left( 1 - \frac{1}{n} \right)^{n-1}
\]
Contestation Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. \( p = \frac{1}{n} \)

**Goals:**
- Low contention
- Low waiting time for all processors

Independent Events A, B:
\[
\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)
\]

\[ \Pr[S(i, t)] = \frac{p}{n-1} \times (1 - \frac{p}{n-1})^{n-1} = \frac{1}{n} \times (1 - \frac{1}{n})^{n-1} \]

\( S(i, t) = \) Event that processor \( i \) succeeds at time \( t = \) At time \( t \), only processor \( i \) accesses
Contestion Resolution

Given $n$ processors and a resource which they all wish to access, design a symmetry breaking access protocol.

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time $t$, each processor accesses resource w.p. $p = 1/n$

$S(i, t) = \text{Event that processor } i \text{ succeeds at time } t = \text{At time } t, \text{ only processor } i \text{ accesses}$

$$\Pr[S(i, t)] = p \times (1 - p)^{n-1} = \frac{1}{n} \times (1 - \frac{1}{n})^{n-1}$$

- $p$: $i$ accesses
- $1 - p$: no other processors access
Contention Resolution

Given \( n \) processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time \( t \), each processor accesses resource w.p. \( p = \frac{1}{n} \)

\[
\Pr[S(i, t)] = \frac{1}{n} \times \left(1 - \frac{1}{n}\right)^{n-1} = \frac{1}{n} \times \left(1 - \frac{1}{n}\right)^{n-1}
\]

\( S(i, t) \) = Event that processor \( i \) succeeds at time \( t \) = At time \( t \), only processor \( i \) accesses

**Fact:** As \( n \) increases,

1. \( (1 - \frac{1}{n})^n \) grows monotonically up to \( 1/e \)
2. \( (1 - \frac{1}{n})^{n-1} \) decreases monotonically down to \( 1/e \)
Contention Resolution

Given \( n \) processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time \( t \), each processor accesses resource w.p. \( p = \frac{1}{n} \)

\[ \Pr[S(i, t)] = \frac{p}{n} \times (1 - p)^{n-1} = \left(\frac{1}{n}\right) \times (1 - \frac{1}{n})^{n-1} \]

\( S(i, t) \) = Event that processor \( i \) succeeds at time \( t \) = At time \( t \), only processor \( i \) accesses

\[ \Pr[S(i, t)] = \frac{p}{n} \times (1 - p)^{n-1} \geq \frac{1}{en} \]

\[ \Pr[S(i, t)] = \left(\frac{1}{n}\right) \times (1 - \frac{1}{n})^{n-1} \leq \frac{1}{2n} \]

**Fact:** As \( n \) increases,

1. \((1 - \frac{1}{n})^n\) grows monotonically up to \( \frac{1}{e} \)
2. \((1 - \frac{1}{n})^{n-1}\) decreases monotonically down to \( \frac{1}{e} \)
Given $n$ processors and a resource which they all wish to access, design a symmetry breaking access protocol.

**Restriction:** Processors can’t communicate  
Simultaneous access blocks the resource

**Protocol:** At time $t$, each processor accesses resource w.p. $p = 1/n$

$S(i, t) =$ Event that processor $i$ succeeds at time $t =$ At time $t$, only processor $i$ accesses

$$Pr[S(i, t)] = p \times (1 - p)^{n-1} = \left(\frac{1}{n}\right) \times \left(1 - \frac{1}{n}\right)^{n-1}$$

$$\geq \left(\frac{1}{en}\right)$$

$$\leq \left(\frac{1}{2n}\right)$$
Contention Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. $p = 1/n$

$S(i, t) = \text{Event that processor } i \text{ succeeds at time } t = \text{At time } t, \text{ only processor } i \text{ accesses}$

$$\Pr[S(i, t)] = p \times (1 - p)^{n-1} = \frac{1}{n} \times (1 - \frac{1}{n})^{n-1}$$

$\geq \frac{1}{en}$

$\leq \frac{1}{2n}$

$F(i, t) = \text{Event that processor } i \text{ has not succeeded after } t \text{ steps}$
Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. p = 1/n

$S(i, t) = \text{Event that processor } i \text{ succeeds at time } t = \text{At time } t, \text{ only processor } i \text{ accesses}

\[
\Pr[S(i, t)] = p \times (1 - p)^{n-1} = \frac{1}{n} \times (1 - \frac{1}{n})^{n-1} \geq \frac{1}{en} \\
\]

$F(i, t) = \text{Event that processor } i \text{ has not succeeded after } t \text{ steps}

\[
\Pr[F(i, t)] = (1 - S(i, 1))(1 - S(i, 2))... (1 - S(i, t)) \leq (1 - \frac{1}{en})^t
\]
Contention Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

Restriction: Processors can’t communicate
Simultaneous access blocks the resource

Protocol: At time t, each processor accesses resource w.p. p = 1/n

\[ S(i, t) = \text{Event that processor } i \text{ succeeds at time } t = \text{At time } t, \text{ only processor } i \text{ accesses} \]

\[ \Pr[S(i, t)] = p \times (1 - p)^{n-1} = \frac{1}{n} \times (1 - 1/n)^{n-1} \geq \frac{1}{en} \]

\[ \Pr[S(i, t)] \geq \frac{1}{en} \leq \frac{1}{2n} \]

\[ F(i, t) = \text{Event that processor } i \text{ has not succeeded after } t \text{ steps} \]

\[ \Pr[F(i, t)] = (1 - \Pr[S(i, 1)])(1 - \Pr[S(i, 2)])...(1 - \Pr[S(i, t)]) \leq (1 - 1/en)^t \]

\[ \Pr[F(i, t)] \leq (1 - 1/en)^t \]

Picking t = en, \[ \Pr[F(i, t)] \leq 1/e \]

Picking t = c. en. ln n, \[ \Pr[F(i, t)] \leq n^{-c} \]
Contention Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol.

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. p = 1/n

**Properties:**
- F(i, t) = Event that processor i has not succeeded after t steps
- Picking t = en, \( \Pr[F(i, t)] \leq 1/e \)
- Picking t = c. en. ln n, \( \Pr[F(i, t)] \leq n^{-c} \)
Contestation Resolution

Given $n$ processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time $t$, each processor accesses resource w.p. $p = 1/n$

**Properties:**
- $F(i, t) = \text{Event that processor } i \text{ has not succeeded after } t \text{ steps}$
- Picking $t = en$, $\Pr[F(i, t)] \leq 1/e$
- Picking $t = c \cdot en \cdot \ln n$, $\Pr[F(i, t)] \leq n^{-c}$

**Fact:** W.p. $1 - 1/n$, all the processors succeed within $2en \cdot \ln n$ rounds
Contestation Resolution

Given $n$ processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time $t$, each processor accesses resource w.p. $p = 1/n$

**Properties:**
- $F(i, t) = \text{Event that processor } i \text{ has not succeeded after } t \text{ steps}$
- Picking $t = en$, $\Pr[F(i, t)] \leq 1/e$
- Picking $t = c \cdot en \cdot \ln n$, $\Pr[F(i, t)] \leq n^{-c}$

**Fact:** W.p. $1 - 1/n$, all the processors succeed within $2en \cdot \ln n$ rounds
- For $t = 2en \cdot \ln n$, $\Pr[F(i, t)] = n^{-2}$ for a fixed $i$
Contention Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can't communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. p = 1/n

**Properties:**

\[ F(i, t) = \text{Event that processor } i \text{ has not succeeded after } t \text{ steps} \]

Picking \( t = en \), \( \Pr[F(i, t)] \leq 1/e \)

Picking \( t = c \cdot en \cdot \ln n \), \( \Pr[F(i, t)] \leq n^{-c} \)

**Fact:** W.p. 1 - 1/n, all the processors succeed within 2en. ln n rounds

For \( t = 2en \cdot \ln n \), \( \Pr[F(i,t)] = n^{-2} \) for a fixed i

**Union Bound:** Any \( A, B \)
\[ \Pr[A \cup B] \leq \Pr[A] + \Pr[B] \]
Contestion Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. p = 1/n

**Properties:**

\[ F(i, t) = \text{Event that processor i has not succeeded after t steps} \]

Picking \( t = en \), \( \Pr[F(i, t)] \leq 1/e \)

Picking \( t = c \cdot en \cdot \ln n \), \( \Pr[F(i, t)] \leq n^{-c} \)

**Fact:** W.p. 1 - 1/n, all the processors succeed within 2en. \ln n rounds

For \( t = 2en \cdot \ln n \), \( \Pr[F(i, t)] = n^{-2} \) for a fixed i

Therefore,

\[
\Pr[\bigcup_{i=1}^{n} F(i, t)] \leq \sum_{i=1}^{n} \Pr[F(i, t)] \leq n \cdot n^{-2} \leq 1/n
\]

**Union Bound:** Any A, B

\( \Pr[A \cup B] \leq \Pr[A] + \Pr[B] \)
Summary: Contention Resolution

Given n processors and a resource which they all wish to access, design a symmetry breaking access protocol

**Restriction:** Processors can’t communicate
Simultaneous access blocks the resource

**Protocol:** At time t, each processor accesses resource w.p. p = 1/n
F(i, t) = Event that processor i has not succeeded after t steps

**Facts:** Picking t = c. en. ln n, Pr[F(i, t)] <= n^-c
WP 1 - 1/n, all processors succeed within 2en ln n rounds

**Facts we learnt:**

**Independent Events** A, B: Pr(A and B) = Pr(A) x Pr(B)

**Union Bound:** For any two events A and B, Pr[A U B] <= Pr[A] + Pr[B]

As n increases,
(1) (1 - 1/n)^n converges monotonically up to 1/e
(2) (1 - 1/n)^n-1 converges monotonically down to 1/e
Randomized Algorithms

- Contention Resolution
- Some Facts about Random Variables
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_ip_i$$
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Examples:**

1. Let $X = 1$ if a fair coin toss comes up heads, 0 otherwise. What is $E[X]$?
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

Examples:

1. Let $X = 1$ if a fair coin toss comes up heads, 0 ow. What is $E[X]$?

2. We are tossing a coin with head probability $p$, tail probability $1 - p$. Let $X =$ #independent flips until first head. What is $E[X]$?
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Examples:**

1. Let $X = 1$ if a fair coin toss comes up heads, 0 ow. What is $E[X]$?
2. We are tossing a coin with head probability $p$, tail probability $1 - p$. Let $X =$ #independent flips until first head. What is $E[X]$?

   $$\Pr[X = j] = p \cdot (1 - p)^{j-1}$$

   head on toss $j$  first $j-1$ tails

   $$E[X] = \sum_{j=1}^{\infty} j \cdot p(1 - p)^{j-1} = \frac{p}{1-p} \cdot \sum_{j=0}^{\infty} j(1 - p)^{j} = \frac{p}{1-p} \cdot \frac{1-p}{p^2} = \frac{1}{p}$$
Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]$
Expectedation

Given discrete random variable X, which takes m values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]$

**Example: Guessing a card**

Shuffle n cards, then turn them over one by one. Guess what the next card is. How many guesses are correct on expectation?
Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]

**Example: Guessing a card**

Shuffle $n$ cards, then turn them over one by one. Guess what the next card is. How many guesses are correct on expectation?

Let $X_i = 1$ if guess $i$ is correct, $0$ otherwise
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_ip_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]$  

**Example: Guessing a card**

Shuffle $n$ cards, then turn them over one by one. Guess what the next card is. How many guesses are correct on expectation?

Let $X_i = 1$ if guess $i$ is correct, 0 otherwise

$$\Pr[X_i = 1] = \frac{1}{n - i + 1}$$
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]$

**Example: Guessing a card**

Shuffle $n$ cards, then turn them over one by one. Guess what the next card is. How many guesses are correct on expectation?

Let $X_i = 1$ if guess $i$ is correct, $0$ otherwise

$$Pr[X_i = 1] = \frac{1}{n - i + 1} \quad E[X_i] = \frac{1}{n - i + 1}$$
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]$

**Example: Guessing a card**

Shuffle $n$ cards, then turn them over one by one. Guess what the next card is. How many guesses are correct on expectation?

Let $X_i = 1$ if guess $i$ is correct, 0 otherwise

$$\Pr[X_i = 1] = \frac{1}{n - i + 1} \quad \quad E[X_i] = \frac{1}{n - i + 1}$$

Expected # of correct guesses = $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$
Expectation

Given discrete random variable $X$, which takes $m$ values $x_i$ w.p. $p_i$, the expectation $E[X]$ is defined as:

$$E[X] = \sum_{i=1}^{m} x_i \cdot Pr[X = x_i] = \sum_{i=1}^{m} x_i p_i$$

**Linearity of Expectation:** $E[X + Y] = E[X] + E[Y]$

**Example: Guessing a card**

Shuffle $n$ cards, then turn them over one by one. Guess what the next card is. How many guesses are correct on expectation?

Let $X_i = 1$ if guess $i$ is correct, 0 otherwise

$$Pr[X_i = 1] = \frac{1}{n - i + 1} \quad E[X_i] = \frac{1}{n - i + 1}$$

Expected # of correct guesses = $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$

What if we insert the selected card into the pile randomly and pull another?