1 Binary Number Systems

1. (one’s complement) Show the operation of \(-10 + (-5)\) in 6-bit one’s complement.

2. (two’s and one’s complements) We have defined and learned the idea of two’s and one’s complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 10. Show the arithmetic of \(-x+y\) where \(x = 216_{10}\) and \(y = 65_{10}\) in the corresponding complement representations using a 6-digit system with base 10.

3. (two’s and one’s complements) We have defined and learned the idea of two’s and one’s complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 8. Show the arithmetic of \(x + y\) where \(x = 120_8\) and \(y = 27_8\) in the corresponding complement representations using a 6-digit system with base 8.

2 Boolean Algebra

1. (expression in sum of products) Express Boolean function 
   \[E(x, y, z) = (x + y + x'z')(x'y' + xy'z')\] in sum-of-products form.

2. (expression in product of sums) Express Boolean function 
   \[E(x, y, z) = [(x'y + x)(x' + y)(y' + z)]'\] in product-of-sums form.

3. (expression in sum of products) Express Boolean function 
   \[E(a, b, c, d) = ab + (cd + bc)' + ad\] in sum-of-products form.

4. (expression in product of sums) Express Boolean function 
   \[E(x, y, z) = [xy'(x'y + z)]'\] in product-of-sums form.

3 Recursive function

1. A frog knows 5 jumping styles (A, B, C, D, E). A, B jump forward by 1 foot, and C, D, E jump forward by 2 feet. Let \(a_i\) denote the number of ways to jump over a total distance of \(i\) feet.
   (a) What is \(a_1, a_2, a_3\)?
   (b) Derive the recursive formula of \(a_n\).
   (c) Find the solution of the recursion.

2. Find the solution of the following recurrence:
   \[
   a_n = -a_{n-1} + a_{n-2} + a_{n-3} \\
   a_0 = 0 \\
   a_1 = 0 \\
   a_2 = 1
   \]

3. Consider the following homogeneous linear recurrence relation:
   \[a_n = 3ra_{n-1} - 3r^2a_{n-2} + r^3a_{n-3}\]. Show that \(a_n = c_1r^n + c_2nr^n + c_3n^2r^n\) satisfies the recurrence relation, where \(c_1, c_2,\) and \(c_3\) are constant coefficients.
4 Pigeonhole principle

1. (points in a circle area) Put 6 points in a plane circle area, prove there are 2 points with distance \( \leq \) radius.
2. (reverse of multiplication) Assume that \( p \) is a prime number. Prove that for any non-zero integer \( a \) with \( 0 < a < p \), there is an integer \( 0 < b < p \) such that \((ab) \equiv 1 \).
3. (seating in a row) 9 people are seated in a row of 12 chairs. Prove that there must be at least three consecutive seats with people in them.

5 Counting

1. (counting numbers) How many zeros do we need to write from 1 to 1000? (For example, we need one zero for each in a set \( \{10, 20, 90, 109, 906\} \), two zeros in a set \( \{100, 300, 900\} \).)
2. (possible routing paths) How many ways to walk from \((0, 0)\) to \((8, 10)\), assuming the streets are all on a grid, and the walking distance must be shortest.
3. (integer linear equation) Find the number of nonnegative integer solutions to \( w + x + y + z = 29 \) with constraints that \( w < 8, x > 1, y < 4, z < 10 \).
4. (integer linear equation) Find the number of nonnegative integer solutions to \( x + y - z = 15 \) with constraints that \( x < 8, y < 9, z < 5 \).
5. (inclusion and exclusion theorem) Prove the inclusion and exclusion theorem when the number of sets is 3, as stated in the following equation.
\[
\left| A \cup B \cup C \right| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|,
\]
where \( |X| \) is the number of elements in set \( X \).
6. (inclusion and exclusion theorem) Prove the inclusion and exclusion theorem when the number of sets is 4, as stated in the following equation.
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|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |B \cap C| - |C \cap D| - |A \cap B \cap C| - |A \cap B \cap D| - |A \cap C \cap D| - |B \cap C \cap D| - |A \cap B \cap C \cap D|
\]