

# 1 Binary Number Systems

1. (one's complement) Show the operation of  $-10 + (-5)$  in 6-bit one's complement.
2. (two's and one's complements) We have defined and learned the idea of two's and one's complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 10. Show the arithmetic of  $-x+y$  where  $x = 216_{10}$  and  $y = 65_{10}$  in the corresponding complement representations using a 6-digit system with base 10.
3. (two's and one's complements) We have defined and learned the idea of two's and one's complements for n-bit binary numbers. Define the corresponding complements using an n-digit system with base 8. Show the arithmetic of  $-x-y$  where  $x = 120_8$  and  $y = 27_8$  in the corresponding complement representations using a 6-digit system with base 8.

# 2 Boolean Algebra

1. (expression in sum of products) Express Boolean function  $E(x, y, z) = (x + y + x'z)'(x'y' + xy'z)$  in sum-of-products form.
2. (expression in product of sums) Express Boolean function  $E(x, y, z) = [(x'y + x)'(x' + y)(y' + z)]'$  in product-of-sums form.
3. (expression in sum of products) Express Boolean function  $E(a, b, c, d) = ab + (cd + bc)' + ad$  in sum-of-products form.
4. (expression in product of sums) Express Boolean function  $E(x, y, z) = [xy'(x'y + z)]'$  in product-of-sums form.

# 3 Recursive function

1. A frog knows 5 jumping styles (A, B, C, D, E). A, B jump forward by 1 foot, and C, D, E jump forward by 2 feet. Let  $a_i$  denote the number of ways to jump over a total distance of  $i$  feet.
  - (a) What is  $a_1, a_2, a_3$ ?
  - (b) Derive the recursive formula of  $a_n$ ?
  - (c) Find the solution of the recursion.
2. Find the solution of the following recurrence:

$$\begin{aligned}a_n &= -a_{n-1} + a_{n-2} + a_{n-3} \\a_0 &= 0 \\a_1 &= 0 \\a_2 &= 1\end{aligned}$$

3. Consider the following homogeneous linear recurrence relation:  
 $a_n = 3ra_{n-1} - 3r^2a_{n-2} + r^3a_{n-3}$ . Show that  $a_n = c_1r^n + c_2nr^n + c_3n^2r^n$  satisfies the recurrence relation, where  $c_1, c_2$ , and  $c_3$  are constant coefficients.

## 4 Pigeonhole principle

1. (points in a circle area) Put 6 points in a plane circle area, prove there are 2 points with distance  $\leq$  radius.
2. (reverse of multiplication) Assume that  $p$  is a prime number. Prove that for any non-zero integer  $a$  with  $0 < a < p$ , there is an integer  $0 < b < p$  such that  $(ab) \% p = 1$ .
3. (seating in a row) 9 people are seated in a row of 12 chairs. Prove that there must be at least three consecutive seats with people in them.

## 5 Counting

1. (counting numbers) How many zeros do we need to write from 1 to 1000? (For example, we need one zero for each in a set  $\{10, 20, 90, 109, 906\}$ , two zeros in a set  $\{100, 300, 900\}$ .)
2. (possible routing paths) How many ways to walk from  $(0, 0)$  to  $(8, 10)$ , assuming the streets are all on a grid, and the walking distance must be shortest.
3. (integer linear equation) Find the number of nonnegative integer solutions to  $w + x + y + z = 29$  with constraints that  $w < 8, x > 1, y < 4, z < 10$ .
4. (integer linear equation) Find the number of nonnegative integer solutions to  $x + y - z = 15$  with constraints that  $x < 8, y < 9, z < 5$ .
5. (inclusion and exclusion theorem) Prove the inclusion and exclusion theorem when the number of sets is 3, as stated in the following equation.  
 $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ , where  $|X|$  is the number of elements in set  $X$ .
6. (inclusion and exclusion theorem) Prove the inclusion and exclusion theorem when the number of sets is 4, as stated in the following equation.  
 $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$