CSE 20 Lecture 15
Analysis: Inclusion-Exclusion Technique

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Analysis: Inclusion – Exclusion Principle

Ex: A bakery makes only M=3 kinds of cookies. Find the number of ways a person can buy r=4 of the cookies.

\[ x+y+z = 4 \] (each variable represents the cookie in that flavor)

\[ C(4+2, 2) = \frac{(6*5)}{2!} = 15 \]

\[ C(n + m-1, m-1), \]

\( n = \#\text{cookies}, \)

\( m = \#\text{kinds of cookies} \)
Cont. (from last slide)

Ex: Buy 2 cookies out of 3 flavors

\[ x + y + z = 2 \]
\[ C(2+2, 2) = C(4,2) \]
\[ = (4*3)/2! = 6 \]
Cont. (from last slide)

Ex: Buy 8 cookies out of 4 kinds of flavors.

\[ x_1 + x_2 + x_3 + x_4 = 8 \]

\[ C(8+3,3) = \frac{11!}{(8!3!)} = \frac{(11*10*9)}{3!} = 165. \]

Ex: Buy 20 cookies out of 5 kinds of flavors.

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 20 \]

\[ C(20+4,4) = \frac{24!}{(20!4!)} = \frac{(24*23*22*21)}{4!} \]
Ex: Possible number of Routing Paths

Assume that the streets are on a grid and the walking distance must be the shortest. Find the number of ways to walk from (0,0) to (6,3).

View from vertical direction:
\[ y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 3 \]
\[ \Rightarrow \binom{3+6}{6} = \binom{9}{6} = \frac{9!}{6!3!} \]

View from horizontal direction:
\[ x_0 + x_1 + x_2 + x_3 = 6 \]
\[ \Rightarrow \binom{6+3}{3} = \binom{9}{6} = \frac{9!}{6!3!} \]
Cont. (from last slide)

Ex: On a grid street, find # shortest paths from (1,3) to (2,5).

Formulations: \( x_1 + x_2 = (5-3) \)

Number of combinations: \( C(2+1,1) = 3!/2! \)

Ex: On a Manhattan street, find # shortest paths from (-1,-2) to (5,8).

\[ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = (8-(-2)) \]

Number of combinations:

\[ C(10+6, 6) = C(16, 6) = 16!/(10!6!) \]
Integer Linear Equation

Ex: Find the number of integer solutions \( x+y+z = 20 \) with \( x \geq 4, \ y \geq 5, \ z \geq 6 \).

Let \( x' = x-4, \ y' = y-5, \ z' = z-6 \),
i.e. \( x = x' + 4, \ y = y' + 5, \ z = z' + 6 \).

Thus, we have an equivalent formulation:
\[
x' + y' + z' = 5 \text{ with } x', y', z' \geq 0.
\]

#Combinations: \( C(5+2, \ 2) = C(7, 2) = \frac{7!}{(5!2!)} = (7 \times 6)/2 = 21 \)
Ex: \( x+y+z = 20 \), \( x<7 \), \( y<8 \), \( z<9 \)

U: no constraints. \( |U| = C(20+2, 2) \)

A: a set of integer solutions with \( x \geq 7 \)

B: a set of integer solutions with \( y \geq 8 \)

C: a set of integer solutions with \( z \geq 9 \)

A: Let \( x'=x-7 \), i.e. \( x=x'+7 \) \( |A| = C(13+2, 2) \)

\( x'+y+z = 13 \)

B: Let \( y'=y-8 \), i.e. \( y=y'+8 \) \( |B| = C(12+2, 2) \)

\( x+y'+z = 12 \)

C: Let \( z'=z-9 \) \( |C| = C(11+2, 2) \)

\( x+y+z' = 11 \)
Cont. (from last slide)

\( A \cap B: x \geq 7, y \geq 8, \)

Let \( x' = x - 7, y' = y - 8 \)

We have \( x' + y' + z = 5. \)

\( |A \cap B| = C(20-7-8+2, 2) \)

Likewise, we can derive \( |A \cap C| = C(20-7-9+2, 2), |B \cap C| = C(20-8-9+2, 2), |A \cap B \cap C| = 0. \)

According to inclusion and exclusion theorem, the number of solutions is

\[
|U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|
\]

= 3