CSE 20 Lecture 13
Analysis: Counting with Pigeonhole Principle

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3.3 Pigeonhole Principle

**Pigeonhole Principle:** If $n$ pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

**Remark:** The principle is obvious. No simpler fact or rule to support or prove it.

**Generalized Pigeonhole Principle:** If $n$ pigeonholes are occupied by $kn + 1$ pigeons, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.
Example 1: Birthmonth

In a group of 13 people, we have 2 or more who are born in the same month.

<table>
<thead>
<tr>
<th># pigeons</th>
<th># holes</th>
<th>At least people born on the same month</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>12</td>
<td>2 or more</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>2 or more</td>
</tr>
<tr>
<td>121</td>
<td>12</td>
<td>11 or more</td>
</tr>
<tr>
<td>65</td>
<td>12</td>
<td>6 or more</td>
</tr>
<tr>
<td>111</td>
<td>12</td>
<td>10 or more</td>
</tr>
<tr>
<td>kn+1</td>
<td>n</td>
<td>k+1</td>
</tr>
</tbody>
</table>
Example 2: Handshaking

Given a group of \( n \) people (\( n > 1 \)), each shakes hands with some (a nonzero number of) people in the group. We can find at least two who shake hands with the same number of people.

Proof:
Number of pigeons (number of people): \( n \)
Number of pigeonholes (range of number of handshakes): \( n-1 \)
Example 3: Cast in theater

A theater performs 7 plays in one season. Five women are each cast in 3 of the plays. Then some play has at least 3 women in its cast.

Number of pigeons (5*3): 15
Number of pigeonholes: 7
k*n+1=2*7+1
3 or more pigeons in the same pigeonhole
Example 4: Pairwise difference

Given 8 different natural numbers, none greater than 14. Show that at least three pairs of them have the same difference.

Try a set: 1, 2, 3, 7, 9, 11, 12, 14
Difference of 12 and 14 = 2.
Same for 9 and 12, 7 and 9, 1 and 3.
In this set, there are four pairs that all have the same difference.

Proof:
Number of pigeons (different pairs: \( \binom{8}{2} = \frac{8 \times 7}{2} \)): 28
Number of pigeonholes (14-1): 13

\( k \times n + 1 = 2 \times 13 + 1 \)

3 or more pigeons in the same pigeonhole
Example 5: Consecutive game plays

A team plays 12 (g) games in a 10 (n)-day period and at least one game a day. We can always find a period of days in which exactly 8 (m) games are played.

Example set:

<table>
<thead>
<tr>
<th>days</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>games</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>s_i</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>t_i</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
</tbody>
</table>

s_i = number of games played from day 1 to day i

\[ t_i = s_i + 8 \]

Number of pigeons (number of symbols, \(2n+2\)): 22

Number of pigeonholes (range, \(g+m+1\)): 21 (from 0 to 20)

\[ s_i \neq s_j, \quad \forall \ i \neq j \]

\[ t_i \neq t_j, \quad \forall \ i \neq j \]

2 or more pigeons in the same pigeon hole

s_i = t_j = s_j + 8, so s_i − s_j = 8, 8 games during day (j+1) ~ i