PRINT your name here: ______________________

- Print your name immediately on the cover page, as well as each page of the exam, in the space provided. Each time you are caught working on a page without your name printed on it, you will lose one point.

- This exam is closed book. You are only allowed to use one page of notes (double sided is fine)

- Your solution will be evaluated both for correctness and clarity. A poorly written solution won’t get full credit even if correct.

- Read all the problems first before start working on any of them, so you can manage your time wisely

- Please, write your answers on the space provided on the front of each page. All answers are fairly short, and the space provided should be enough. You can use the back side of the pages as scratch paper.

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1 True or False [10 points]

For each of the following statements, mark if the statement is true or false. (No justification is required. You get 1 point for each correct answer, and 0 points for each blank or wrong one.)

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<td>P(∅) ⊆ {∅}</td>
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2 Sets [8 points]

Let $A$, $B$ and $C$ be the sets

- $A = \{x \in \mathbb{R} \mid -5 \leq x \leq 0\}$
- $B = \{x \in \mathbb{R} \mid -1 < x < 1\}$
- $C = \{x \in \mathbb{R} \mid -4 \leq x < 3\}$

For each of the following expressions, complete the definition of the corresponding set:

- $A \cup B \cup C = \{x \in \mathbb{R} \mid -5 \leq x < 3\}$.

- $A^c \cap B^c = \{x \in \mathbb{R} \mid x < -5 \lor x \geq 1\}$.

- $(A \cup B)^c \cap C = \{x \in \mathbb{R} \mid 1 \leq x < 3\}$.

- $A \cap B \cap C^c = \{x \in \mathbb{R} \mid false\}$. 

3 Venn Diagrams [9 points]

Draw Venn diagrams to describe sets $A$, $B$ and $C$ satisfying the given conditions.

(a) $A \subseteq B$ and $A \cap C = \emptyset$

(b) $A \cup B \subseteq C$

(c) $A \cap B \subseteq C$
4 Counterexample [9 points]

For each of the following statements, give a counterexample showing that the statement is wrong.

(a) If $A \subseteq B$ and $A^c \subseteq C$, then $A \subset B \cap C$.

$$U = \{1, 2, 3\}, A = \{1\}, B = \{1, 2\} \text{ and } C = \{2, 3\}.$$ 

(b) For all sets $A$ and $B$, either $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$.

$$A = \{1, 2\}, B = \{2, 3\}.$$ 

(c) $A \cup (B - C) = (A \cup B) - C$

$$A = \{1\}, B = \{1\}, C = \{1\}.$$
5 Proof 1 [10 points]

Prove using the element method that for any sets $A$, $B$ and $C$,
\[(A - B) \cap (C - B) = (A \cap C) - B\]

Let $A$, $B$, $C$ be any particular sets.

\[((A - B) \cap (C - B) \subseteq (A \cap C) - B\]: Suppose $x \in (A - B) \cap (C - B)$. By definition of intersection, $x \in (A - B)$ and $x \in (C - B)$. Also, by definition of difference, $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$. Since, $x \in A$ and $x \in C$, then by definition of intersection, we have $x \in A \cap C$. Also, since $x \notin B$ and from definition of difference, $x \in (A \cap C) - B$. [Thus $(A - B) \cap (C - B) \subseteq (A \cap C) - B$ by definition of subset]

\[((A \cap C) - B \subseteq (A - B) \cap (C - B))\]: Suppose $x \in (A \cap C) - B$. By definition of difference, $x \in (A \cap C)$ and $x \notin B$. Also, by definition of intersection, $x \in A$ and $x \in C$ and $x \notin B$. Since, $x \in A$ and $x \notin B$, then by definition of difference, we have $x \in A - B$. Also, since, $x \in C$ and $x \notin B$, then by definition of difference, we have $x \in C - B$. Next we have, $x \in (A - B) \cap (C - B)$ by definition of intersection. [Thus $(A \cap C) - B \subseteq (A - B) \cap (C - B)$ by definition of subset]

Since both subset containments have been proved, $(A - B) \cap (C - B) = (A \cap C) - B$ by definition of set equality.
6 Induction [8 points]

Complete the proof of the following theorem.

Theorem 1 For any \( n \) sets \( A_1, A_2, \ldots, A_n \), the sets \( B_1, B_2, \ldots, B_n \) defined by the rules \( B_1 = A_1 \) and \( B_i = A_i \setminus (\bigcup_{j<i} A_j) \) (for all \( i > 1 \)) form a partition of \( A = \bigcup_{i=1}^{n} A_i \).

Proof: Remember that \( B_1, \ldots, B_n \) is a partition of \( A \) if

1. \( \bigcup_{i=1}^{n} B_i = A \), and
2. For all \( i \neq j \), \( B_i \cap B_j = \emptyset \).

We prove that the statement in the theorem is true for every \( n \geq 1 \) by induction on \( n \).

Base case \((n = 1)\) When \( n = 1 \), we have only one set \( A_1 \). It follows that \( A_1 = B_1 = A \), and \( \{B_1\} \) is a trivial partition of \( A \).

Inductive case \((n > 1)\) Assume that the statement in the theorem holds true for any \( n-1 \) sets \( A_1, \ldots, A_{n-1} \), and let \( A_1, \ldots, A_n \) be \( n \) sets. ...

We know from inductive hypothesis,

1. \( \bigcup_{i=1}^{n-1} B_i = \bigcup_{i=1}^{n-1} A_i \), and
2. \( \forall i \neq j, B_i \cap B_j = \emptyset, 1 \leq i, j \leq (n-1) \).

We must show that,

1. \( \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \), and
2. \( \forall i \neq j, B_i \cap B_j = \emptyset, 1 \leq i, j \leq n \).
1). Let, \( M = \bigcup_{i=1}^{n-1} A_i \), then the left hand side of the first part is

\[
\bigcup_{i=1}^{n} B_i
= (\bigcup_{i=1}^{n-1} B_i) \cup B_n
= (\bigcup_{i=1}^{n-1} A_i) \cup (A_n - (\bigcup_{i=1}^{n-1} A_i))
\]

[From, inductive hypothesis and definition of \( B_n \)]
\[
= M \cup (A_n - M)
= M \cup (A_n \cap M^c)
= (M \cup A_n) \cap (M \cup M^c)
= (M \cup A_n) \cap \mathcal{U}
= (\bigcup_{i=1}^{n-1} A_i) \cup A_n
= (\bigcup_{i=1}^{n} A_i)
\]

2). We need to show \( \forall i \neq j, B_i \cap B_j = \emptyset, 1 \leq i, j \leq n \). More specifically, \( \forall i < n, B_n \cap B_i = \emptyset \), as the other pairs are true from inductive hypothesis.

Let \( B_i \) be any particular set such that \( i < n \). [Proof by contradiction]: Suppose \( B_n \cap B_i \neq \emptyset \). Therefore, \( \exists x \in B_n \cap B_i \). That is, \( x \in B_n \) and \( x \in B_i \) by definition of intersection. Next we have, \( x \in A_n \) and \( x \notin \bigcup_{i=1}^{n-1} A_i \) and \( x \in A_i \) and \( x \notin \bigcup_{j<i} A_j \) by definition of \( B_n \) and \( B_i \). In particular, \( x \notin A_i \) since \( x \notin \bigcup_{i=1}^{n-1} A_i \) and \( i < n \). However, \( x \in A_i \) and \( x \notin A_i \) is a contradiction. Therefore the supposition is wrong and \( B_n \cap B_i = \emptyset \). Therefore, \( \forall i \neq j, B_i \cap B_j = \emptyset, 1 \leq i, j \leq n \).