Section 3.4

31. a) Suppose \( m \) and \( n \) are integers.
Case 1 (both \( m \) and \( n \) are even): Therefore, there exists integers \( k \) and \( l \) such that \( m = 2k \) and \( n = 2l \). Now, 
\[
(m + n) = 2(k + l) \text{ is even and } (m - n) = 2(k - l) \text{ is even.}
\]
Case 2 (both \( m \) and \( n \) are odd): Therefore, there exists integers \( k \) and \( l \) such that \( m = 2k+1 \) and \( n = 2l+1 \).
Now, 
\[
(m + n) = 2(k + l + 1) \text{ is even and } (m - n) = 2(k - l) \text{ is even.}
\]
Case 3 (one of \( m \) and \( n \) is even and the other is odd):
- Case a (\( m \) be even and \( n \) be odd): Therefore, there exists integers \( k \) and \( l \) such that \( m = 2k \) and \( n = 2l+1 \). Now, 
\[
(m + n) = 2(k + l) + 1 \text{ is odd and } (m - n) = 2(k - l) - 1 \text{ is odd.}
\]
- Case b (\( m \) be odd and \( n \) be even): Therefore, there exists integers \( k \) and \( l \) such that \( m = 2k+1 \) and \( n = 2l \). Now, 
\[
(m + n) = 2(k + l) + 1 \text{ is odd and } (m - n) = 2(k - l) + 1 \text{ is odd.}
\]
Thus, in all three possible cases, either both \( m+n \) and \( m-n \) are even or both are odd. [QED]

31. b) Given, \( m^2 - n^2 = (m - n)(m + n) = 56 \). Now, from unique factorization theorem we know that 
\( 56 = 2^3 \cdot 7 \) and this is factorization is unique. Thus, we can write 56 in the following different product of 2 
positive integers; \( 56 = 7 \cdot 8 = 1 \cdot 4 \cdot 4 = 2 \cdot 2 \). From part 31 (a), we know that either both \( m+n \) and \( m-n \) 
are even or both are odd. So, the only solution are \( m+n= 28 \), \( m-n = 2 \) and \( m+n= 14 \), \( m-n = 4 \). This gives 
either \( m = 15 \) and \( n = 13 \) or \( m = 9 \) and \( n = 5 \) as the only solutions.

31. c) Given, \( m^2 - n^2 = (m - n)(m + n) = 88 \). Now, from unique factorization theorem we know that 
\( 88 = 2^3 \cdot 11 \) and this is factorization is unique. Thus, we can write 88 in the following different product of 2 
positive integers; \( 88 = 88 \cdot 1 = 11 \cdot 8 = 22 \cdot 4 = 44 \). From part 31 (a), we know that either both \( m+n \) and \( m-n \) 
are even or both are odd. So, the only solution are \( m+n= 44 \), \( m-n = 2 \) and \( m+n= 22 \), \( m-n = 4 \). This gives 
either \( m = 23 \) and \( n = 21 \) or \( m = 13 \) and \( n = 9 \) as the only solutions.

35. Suppose \( n \) is any integer. By quotient-remainder theorem with \( d = 2 \), \( n \) is either even or odd.
Case 1 (\( n \) is even): In this case \( n = 2k \) for some integer \( k \), and 
\[
n^4 = (2k)^4 = 16k^4 = 8(2k^4).
\]
Hence \( n^4 = 8m \) where \( m \) is an integer.
Case 2 (\( n \) is odd): In this case \( n = 2k+1 \) for some integer \( k \), and 
\[
n^4 = (2k+1)^4 = 8(2k^4 + 4k^3 + 3k^2 + k) + 1.
\]
Hence \( n^4 = 8m + 1 \) where \( m \) is an integer.
Thus in both cases \( n^4 = 8m \) or \( n^4 = 8m + 1 \) for some integer \( m \). [QED]
25. Suppose $x$ is any particular real number. Let $n = \lfloor x/2 \rfloor$. Then by definition of floor, $n \leq x/2 < n + 1$.

Case 1 (n is even): In this case $n/2$ is an integer, and we divide all parts of the inequality by 2 to obtain $n/2 \leq x/4 < n/2 + 1/2$. But $(n/2 + 1/2) < (n/2 + 1)$. Hence $n/2 \leq x/4 < n/2 + 1$ and because $n/2$ is integer, and so by definition of floor $\lfloor x/4 \rfloor = n/2$. Now, $\lfloor \lfloor x/2 \rfloor/2 \rfloor = \lfloor n/2 \rfloor = n/2 = \lfloor x/4 \rfloor$.

Case 2 (n is odd): In this case $(n-1)/2$ is an integer, and by Theorem 3.5.2 $\lfloor n/2 \rfloor = (n-1)/2$. We divide all parts of the inequality by 2 to obtain $n/2 \leq x/4 < n/2 + 1/2$. But $(n/2 + 1/2) = ((n-1)/2 + 1)$ and $n/2 > (n-1)/2$. Hence $(n-1)/2 \leq x/4 < (n-1)/2 + 1$ and because $(n-1)/2$ is integer, and so by definition of floor $\lfloor x/4 \rfloor = (n-1)/2$. Now, $\lfloor \lfloor x/2 \rfloor/2 \rfloor = \lfloor n/2 \rfloor = (n-1)/2 = \lfloor x/4 \rfloor$.

Thus in both cases, $\lfloor \lfloor x/2 \rfloor/2 \rfloor = \lfloor x/4 \rfloor$. [QED]
Section 3.6

12. [By contradiction] Suppose not. That is, ∃ an integer n such that 4|(n^2 − 2). Therefore, n^2 − 2 = 4m for some integer m. Then n^2 = 4m + 2 = 2(2m + 1), and so n^2 is even. Hence, n is even. Therefore, n = 2k for some integer k. Now, 4k^2 = 4m + 2. Dividing by 2 gives 2k^2 = 2m + 1. Since k^2 is an integer, this implies 2m + 1 is even, but since m is an integer, 2m + 1 is odd. This is a contradiction.